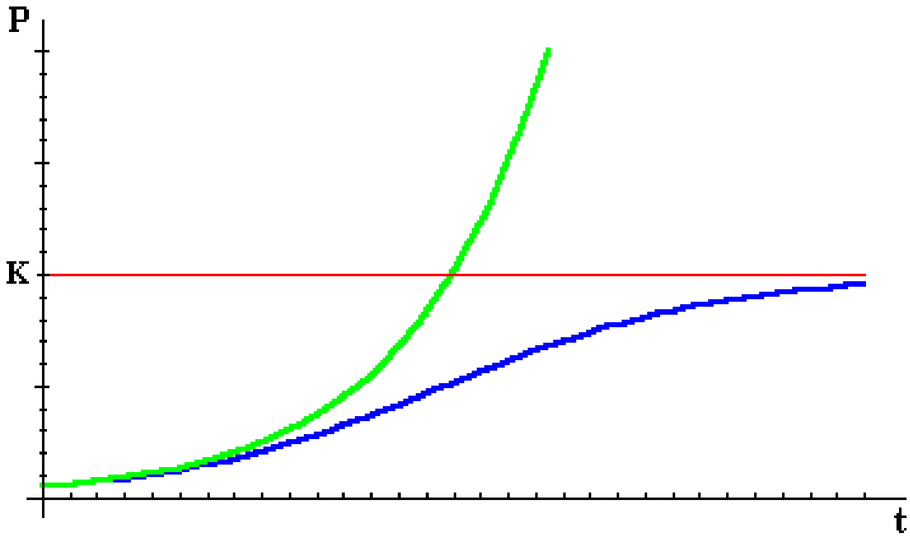
5.1 Logistic Functions

Logistic Growth Model
One of the best known examples of logistic growth is the classic study of the growth of a yeast culture. You may find this example familiar to exponential functions (i.e. the bacteria growing in a petri dish). There are some key features of logistic functions that we should note concerning the graph:



- **Initial rapid growth rate:** Note that if we restrict our attention to the blue curve over about the first half of the time, then we can observe that it grows seemingly exponential; after about the halfway mark the curve levels out and achieves a **limiting value**.

- **Point of inflection:** The graphs has a point of inflection. This is the steepest part of the graph, so the inflection point represents the time of most rapid growth. Before the inflection point, growth is almost exponential.

- **Eventual declining growth rate and limiting value** As stated in the first bullet, the limiting value is represented by the red line, and denoted as the value K, this is known as the *carrying capacity* of the population.

Theory of Maximal Sustainable Yield
In the management of a harvested population (as found in a marine fishery), an important problem is to determine at what level to maintain the population in order to sustain a maximum harvest. This is the problem of optimum yield. To get the best harvest, it would make sense to determine when the population is growing the fastest (i.e. the inflection point). With advanced mathematics, we can determine that the inflection point of a logistic function occurs at one-half of the carrying capacity.

Formula for the Logistic Model

"Deriving the formula for logistic growth requires techniques beyond the scope of this text, ad we simply state the result. The formula for a logistic model is

$$N = \frac{K}{1 + be^{-rt}}.$$

Here K is the carrying capacity. It is the limiting value of N, so the point of inflection occurs at the level $N = K/2$. *Note: this is an output value, so if asked* ***when*** *the inflection point occurs, we must use crossing graphs method to solve.* The constant b is determined by

$$b = \frac{K}{N(0)} - 1.$$

The constant r is the intrinsic exponential growth rate. In absence of limiting factors, growth would be exponential according to the formula $N = N(0)e^{rt}$. The value r is measured units of time. The corresponding growth factor for this exponential function is $a = e^r$, which we can solve for r and get that $r = \ln a$.

Key Ideas: Logistic Model

- Initial growth is rapid, nearly exponential.
- The inflection point represents the time of most rapid growth.
- After the inflection point, the growth rate declines, reaches the limiting value.
- The inflection point occurs at half of the carrying capacity. This is the level of maximum growth.
- The equation for a logistic model is $N = \dfrac{K}{1 + be^{-rt}}$
- The constant K is the carrying capacity. It is the limiting value of N. The inflection point occurs as $N = K/2$.
- The constant b is determined by $b = \dfrac{K}{N(0)} - 1$.
- In the absence of a limiting value, the value of r is found by $r = \ln a$.