On the Universal Enveloping Algebra: Including the Poincaré-Birkhoff-Witt Theorem

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In this paper we aim to provide the reader with an expository look at several important results in the study of finite dimensional Lie algebras. Topics discussed in this text include the construction of the Universal Enveloping Algebra and a famous result of said algebra given the name the Poincaré-Birkhoff-Witt Theorem.

1 Universal Enveloping Algebra

The universal enveloping algebra, denoted $\mathfrak{U}$, is an essential tool for studying representations and more generally for studying homomorphisms of a Lie algebra $\mathfrak{g}$ into an associative algebra with an identity element [Jac, 1979]. Our motivation for constructing $\mathfrak{U}$ is as follows: we wish to view $\mathfrak{g}$ as an associative algebra, namely $\mathfrak{U}$, via the representations of $\mathfrak{U}$. This result is obtained from an important property of $\mathfrak{U}$ which states that $\mathfrak{g}$ is isomorphic to a subalgebra of $\mathfrak{U}(\mathfrak{g})$. From this isomorphism we get a faithful representation for every $\mathfrak{g}$.

1.1 Construction of the Universal Enveloping Algebra

Definition 1.1. An *associative algebra* A is a vector space V over a field $\mathbb{F}$ which contains an associative, bilinear vector product $\cdot : V \times V \to V$. If there is some element $1 \in V$ such that $1 \cdot a = 1 = a \cdot 1$ for every $a \in A$, then A is unital, or "has unit".

Definition 1.2. [Hum, 1997] Let $\mathfrak{g}$ be a Lie algebra over an arbitrary field $\mathbb{F}$. The *universal enveloping algebra* of $\mathfrak{g}$ is a pair $(\mathfrak{U}(\mathfrak{g}), i)$, which satisfy the following:

(i) $\mathfrak{U}(\mathfrak{g})$ is an associative algebra with unit over $\mathbb{F}$.

(ii) $i : \mathfrak{g} \to \mathfrak{U}(\mathfrak{g})$ is linear and $i([x, y]) = i(x)i(y) - i(y)i(x)$, for all $x, y \in \mathfrak{g}$.

(iii) (Universal Property) For any associative algebra A with unit over $\mathbb{F}$ and for any linear map $j : \mathfrak{g} \to A$ satisfying $j([x, y]) = j(x)j(y) - j(y)j(x)$ for each $x, y \in \mathfrak{g}$, there exists a unique homomorphism of algebras $\theta : \mathfrak{U}(\mathfrak{g}) \to A$ such that $\theta \circ i = j$

Moreover, we can say that the following diagram commutes.

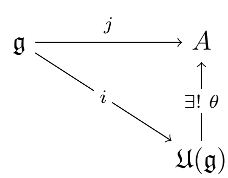

Figure 1: Universal Property

Since $\mathfrak{g}$ is any Lie algebra there is no guarantee that $\mathfrak{g}$ has associative multiplication. Note that the Lie bracket is not necessarily the commutator, however, applying i to the bracket of any two $x, y \in \mathfrak{g}$ must give the commutator of $i(x)$ and $i(y)$. As an aside we should note that Definition 1.2 does not require $\mathfrak{g}$ to be of finite dimension or over a field with a particular characteristic. This leaves us with a possible construction of $\mathfrak{U}(\mathfrak{g})$ for which $\mathfrak{g}$ is infinite dimensional.

Theorem 1.1 (Uniqueness and Existence of $\mathfrak{U}(\mathfrak{g})$)**.** *If* $\mathfrak{g}$ *is any Lie algebra over an arbitrary field* $\mathbb{F}$*, then* $(\mathfrak{U}(\mathfrak{g}), i)$ *exists and is unique, up to isomorphism.*

Proof. (Uniqueness) We prove this in the normal convention in that we suppose that the Lie algebra $\mathfrak{g}$ has two universal enveloping algebras $(\mathfrak{U}(\mathfrak{g})), i)$ and $(\mathfrak{B}(\mathfrak{g}), i')$. By definition, for each associative $\mathbb{F}$-algebra A there exists a unique homomorphism $\varphi_A : \mathfrak{U}(\mathfrak{g}) \to A$. In particular, since $\mathfrak{B}(\mathfrak{g})$ is an associative $\mathbb{F}$-algebra, we have a unique homomorphism of algebras $\phi : \mathfrak{U}(\mathfrak{g}) \to \mathfrak{B}(\mathfrak{g})$. Moreover, we can, by similar logical progression, reverse the roles of $\mathfrak{U}$ and $\mathfrak{B}$; then there must exist a unique homomorphism of algebra $\psi : \mathfrak{B}(\mathfrak{g}) \to \mathfrak{U}(\mathfrak{g})$. Then $\phi \circ \psi = 1_{\mathfrak{U}(\mathfrak{g})}$ and $\psi \circ \phi = 1_{\mathfrak{B}(\mathfrak{g})}$, which implies that ϕ is a bijection. However, ϕ was already a unique homomorphism, therefore it is an isomorphism. Thereby making $(\mathfrak{U}(\mathfrak{g}), i)$ unique, up to isomorphism.

(Existence) The proof of existence requires the use of the tensor algebra, $\mathcal{T}$, which requires extensive background development. And so, we will omit a proof of the existence here, but one can be found in Chapter 17 of [Hum, 1997]. □

Remark 1. Theorem 1.1 reveals to us that $\mathfrak{U}(\mathfrak{g})$ of $\mathfrak{g}$ can be viewed as the *maximal* associative algebra over an arbitrary field with unity generated by $\mathfrak{g}$ satisfying the relation $xy - yx = [x, y]$ for $x, y \in \mathfrak{g}$ [Hon, 2002].

2 Poincaré-Birkhoff-Witt Theorem

Depending upon the textbook from which you are studying, there are different variations of what the author calls the Poincaré-Birkhoff-Witt Theorem (or PBW Theorem). For the purpose of this paper, we will use the formulation of the theorem found in Chapter 1 of [Hon, 2002]. It is interesting, however, to compare how different authors state the PBW Theorem. For example, in [Hum, 1997] the PBW Theorem is defined as an isomorphism between a *symmetric* algebra and a *graded* associative algebra. Moreover, the way the the same theorem is stated in this paper is really a collection of two corollaries of what Humphreys states the PBW-Theorem to be.

2.1 Poincaré-Birkhoff-Witt Theorem

Theorem 2.1 (Poincaré-Birkhoff-Witt Theorem)**.** *[Hon, 2002]*

(i) The map $i : \mathfrak{g} \to \mathfrak{U}(\mathfrak{g})$ *is injective.*

(ii) Let $\{x_\alpha | \alpha \in \Omega\}$ *be an ordered basis of* $\mathfrak{g}$*. Then, all the elements of the form* $x_{\alpha_1} x_{\alpha_2} \cdots x_{\alpha_n}$ *satisfying* $\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_n$ *together with 1 form a basis of* $\mathfrak{U}(\mathfrak{g})$*.*

A proof of the PBW Theorem is left for the interested reader as it is very involved and requires the use of tensor products and several other algebras that have a universal property. A detailed proof can be found in Chapter V of [Jac, 1979].

Part (1) of the Theorem 2.1 shines some light on how we can identify each $g \in \mathfrak{g}$ with $i(g) \in \mathfrak{U}(g)$, thereby allowing us to think of $\mathfrak{U}(\mathfrak{g})$ as a larger algebra "enveloping" $\mathfrak{g}$. The next example shows how we can construct a basis of $\mathfrak{U}(\mathfrak{g})$ using the PBW Theorem. Bases of these type are often called PBW-type bases.

Definition 2.1 (Polynomial Algebra)**.** Let $\mathbb{F}$ be a field. The *polynomial algebra* on n indeterminates $X_1, X_2, \ldots, X_n$ is the algebra that is spanned by all the linear combinations over $\mathbb{F}$ of products of the commuting variables $X_i, 1 \leq i \leq n$. This algebra is denoted $\mathbb{F}[X_i]$.

Definition 2.2 (Symmetric Algebra)**.** The *symmetric algebra* $S(V)$ on a vector space V over a field $\mathbb{F}$ is the free commutative unital associative algebra over $\mathbb{F}$ containing V.

Lemma 2.2. *If* $\mathbb{F}[X_i]$ *is a polynomial algebra and* $S(V)$ *is a symmetric algebra, then* $\mathbb{F}[X_i] \cong S(V)$.

Example 2.1. Let $\mathfrak{g}$ be an abelian Lie algebra of dimension 2 with basis $\{x_1, x_2\}$ over the field $\mathbb{F}$. We know that the bracket $[x_1, x_2] = 0$. So defining the relations of the elements in the basis to be $X_1X_2 - X_2X_1 = 0$, then by Theorem 2.1, we know that all the elements of the form $X_1^a X_2^b$ where $a, b \in \mathbb{Z}_{\geq 0}$ together with 1 form a basis of $\mathfrak{U}(\mathfrak{g})$. But since the relationship yields symmetry of the elements under multiplication, we have that $\mathfrak{U}(\mathfrak{g})$ is symmetric and therefore isomorphic to the polynomial algebra of two variables by Lemma 2.2.

We can extend this to the n-dimensional case for an abelian Lie algebra.

Example 2.2. Let $\mathfrak{g}$ be an abelian Lie algebra of dimension n with a basis $\{x_1, x_2 \ldots, x_n\}$ over the field $\mathbb{F}$. Again, because $\mathfrak{g}$ is abelian, we have that $\forall_{1 \leq i \leq j \leq n}\ [x_i, x_j] = 0$. This tells us that the basis elements of $\mathfrak{U}(\mathfrak{g})$ have the relationship that $\forall_{1 \leq i \leq j \leq n}\ X_iX_j - X_jX_i = 0$. So, the elements in $\mathfrak{U}(\mathfrak{g})$ form a symmetric algebra that is isomorphic to the polynomial algebra of n variables. This is inductively extended from the two dimensional case. So, in this we can view all n-dimensional Lie algebras as a polynomial algebra.

Example 2.3. Consider the following list of matrices:

$$\underbrace{\begin{pmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 0\end{pmatrix}}_{X_1}, \underbrace{\begin{pmatrix}0 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -1\end{pmatrix}}_{X_2}, \underbrace{\begin{pmatrix}0 & 1 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}}_{X_3}, \underbrace{\begin{pmatrix}0 & 0 & 1\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}}_{X_4},$$

$$\underbrace{\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\end{pmatrix}}_{X_5}, \underbrace{\begin{pmatrix}0 & 0 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}}_{X_6}, \underbrace{\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 1 & 0 & 0\end{pmatrix}}_{X_7}, \underbrace{\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 1 & 0\end{pmatrix}}_{X_8}. \quad (1)$$

Let $X_1, X_2, X_3, X_4, X_5, X_6, X_7, X_8$ be a basis for $\mathfrak{sl}(3, \mathbb{F})$. Then the elements of the form $x_1^{\alpha_1}, x_2^{\alpha_2}, x_3^{\alpha_3}, x_4^{\alpha_4}, x_5^{\alpha_5}, x_6^{\alpha_6}, x_7^{\alpha_7}, x_8^{\alpha_8}$ with $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8 \in$

$\mathbb{Z}_{\geq 0}$ form a basis of the universal enveloping algebra $\mathfrak{U}(\mathfrak{sl}(3,\mathbb{F}))$. Consider the triangular decomposition of $\mathfrak{sl}(3,\mathbb{F})$,

$$\mathfrak{sl}(3,\mathbb{F}) = \mathfrak{g}^- \oplus \mathfrak{h} \oplus \mathfrak{g}^+,$$

where $\mathfrak{g}^- = \text{span}\{X_6, X_7, X_8\}$, $\mathfrak{h} = \text{span}\{X_1, X_2\}$, $\mathfrak{g}^+ = \text{span}\{X_3, X_4, X_5\}$ gives us three subalgebras of $\mathfrak{U}(\mathfrak{sl}(3,\mathbb{F}))$. So by Theorem 2.1 $\mathfrak{sl}(3,\mathbb{F})$ is a subspace of $\mathfrak{U}(\mathfrak{sl}(3,\mathbb{F}))$. Thus, $\mathfrak{U}(\mathfrak{sl}(3,\mathbb{F}))$ will contain $\mathfrak{U}(\mathfrak{g}^-)$ which contains all polynomials in $\mathfrak{g}^-$. Similarly, we will get that $\mathfrak{U}(\mathfrak{sl}(3,\mathbb{F}))$ will contain all polynomials in $\mathfrak{h}$ and $\mathfrak{g}^+$. Also, we will get $\mathfrak{U}(\mathfrak{sl}(3,\mathbb{F}))$ contains all products of these elements. We can calculate the commutator bracket for the elements in the basis of $\mathfrak{sl}(3,\mathbb{F})$ and force this relationship on the elements in the basis of $\mathfrak{U}(\mathfrak{sl}(3,\mathbb{F}))$. Let each $x_i, 1 \leq i \leq 8$ abide by the following relationships:

Table of Relations

$x_1x_2 - x_2x_1 = 0$	$x_2x_3 - x_3x_2 = -x_3$	$x_3x_4 - x_4x_3 = 0$
$x_1x_3 - x_3x_1 = 2x_3$	$x_2x_4 - x_4x_2 = x_4$	$x_3x_5 - x_5x_3 = x_4$
$x_1x_4 - x_4x_1 = x_4$	$x_2x_5 - x_5x_2 = 2x_5$	$x_3x_6 - x_6x_3 = x_1$
$x_1x_5 - x_5x_1 = -x_5$	$x_2x_6 - x_6x_2 = x_6$	$x_3x_7 - x_7x_3 = -x_8$
$x_1x_6 - x_6x_1 = -2x_6$	$x_2x_7 - x_7x_2 = -x_7$	$x_3x_8 - x_8x_3 = 0$
$x_1x_7 - x_7x_1 = -x_7$	$x_2x_8 - x_8x_2 = -2x_8$	
$x_1x_8 - x_8x_1 = x_8$		
$x_4x_5 - x_5x_4 = 0$	$x_5x_6 - x_6x_5 = 0$	$x_6x_7 - x_7x_6 = 0$
$x_4x_6 - x_6x_4 = -x_5$	$x_5x_7 - x_7x_5 = x_6$	$x_6x_8 - x_8x_6 = -x_7$
$x_4x_7 - x_7x_4 = x_1 + x_2$	$x_5x_8 - x_8x_5 = x_2$	$x_7x_8 - x_8x_7 = 0$
$x_4x_8 - x_8x_4 = x_3$		

Then $\{x_1^{\alpha_1}x_2^{\alpha_2}x_3^{\alpha_3}x_4^{\alpha_4}x_5^{\alpha_5}x_6^{\alpha_6}x_7^{\alpha_7}x_8^{\alpha_8} | \alpha_i \in \mathbb{Z}_{\geq 0}\}$ is a basis for $\mathfrak{U}(\mathfrak{sl}(3,\mathbb{F}))$.

It is interesting to note that the choice of ordering the basis elements is arbitrary. Up to a different labeling, the PBW-type basis is the same. In construction of these types of bases, the ordering of the basis is imposed, rather than a specific ordering is required.

2.2 Representations of $\mathfrak{U}(\mathfrak{g})$

Definition 2.3. A *representation* of an associative algebra on a vector space V is an algebra homomorphism $\varphi : A \to \text{End } V$.

Like in the case of a Lie algebra, a representation of an associative algebra over a field with unity on a vector space defines a module structure on the vector space and vice versa.

Theorem 2.3. *A representation of $\mathfrak{g}$ can be extended naturally to a representation of $\mathfrak{U}(\mathfrak{g})$. If we let φ be a Lie algebra homomorphism and $\bar{\varphi}$ be an associative algebra homomorphism, then the following diagram commutes. Note: By "restrict" we mean that we are only considering the elements from End(V) for which $[a, b] = ab - ba$ holds.*

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\varphi} & gl(V) \\ \Big\downarrow{\scriptstyle \text{Univ. Prop}} & & \Big\uparrow{\scriptstyle \text{Restrict}} \\ \mathfrak{U}(\mathfrak{g}) & \xrightarrow{\bar{\varphi}} & \text{End}(V) \end{array}$$

Proof. As with representations of Lie algebras, a representation of an associative algebra over a vector space defines a module structure on the vector space, and vice versa. To this end, consider a $\mathfrak{g}$-module, say $\mathcal{V}$, on V, and let $g_1 g_2 g \cdots g_n$ be an element from $\mathfrak{U}(\mathfrak{g})$. We can define the action of $\mathfrak{U}(\mathfrak{g})$ on V by

$$(g_1 g_2 g_3 \cdots g_n) \cdot v = g_1 \cdot ((g_2 g_3 \cdots g_n) \cdot v) = \cdots = g_1 \cdot (g_2 \cdot (g_3 \cdots (g_n \cdot v)))$$

for all $g_1, g_2, \ldots, g_n \in \mathfrak{g}, v \in V$. Since $\mathfrak{U}(\mathfrak{g})$ is generated by $\mathfrak{g}$ (see Remark 1), $g_1 \cdot (g_2 \cdot (g_3 \cdots (g_n \cdot v)))$ will generate $\mathfrak{U}(\mathfrak{g})$ such that $\mathcal{V}$ must also be a $\mathfrak{U}(\mathfrak{g})$-module.

Now, suppose $\mathcal{V}$ is a $\mathfrak{U}(\mathfrak{g})$-module. Since $\mathfrak{g}$ can be identified by elements in $U(\mathfrak{g})$ by the injective mapping we get from part (1) of the PBW theorem, then $\mathcal{V}$ is also a $\mathfrak{g}$-module. Moreover, we have shown that $\mathcal{V}$ can be treated as a $\mathfrak{U}(\mathfrak{g})$-module and $\mathfrak{g}$-module simultaneously; thus, there is a natural extension from representations of $\mathfrak{g}$ to representations of $\mathfrak{U}(\mathfrak{g})$ and vice versa. □

An alternate way of wording Theorem 2.3 is given in [Erd, 2006] and written below.

Theorem 2.4. *[Erd, 2006] Let $\mathfrak{g}$ be a Lie algebra and let $\mathfrak{U}(\mathfrak{g})$ be its universal enveloping algebra. There is a bijective correspondence between $\mathfrak{g}$-modules and $\mathfrak{U}(\mathfrak{g})$-modules.*

The proof given in [Erd, 2006] uses the authors construction of $\mathfrak{U}(\mathfrak{g})$, which differs at length from the one given in Definition 1.2 of this paper. As where we have defined $\mathfrak{U}(\mathfrak{g})$ by its universal property, Erdmann and Wildon have not. Therefore, this result to them proves that $\mathfrak{U}(\mathfrak{g})$ has a universal property as where our construction of $\mathfrak{U}(\mathfrak{g})$ imposes this property on the associative algebra. It should also be noted that this bijective correspondence between modules gives us a faithful representation from $\mathfrak{g}$ to $\mathfrak{U}(\mathfrak{g})$. So when we consider the universal enveloping algebra as a representation of $\mathfrak{g}$, there is no collapse of any important information pertaining to $\mathfrak{g}$.

3 Applications of $\mathfrak{U}(\mathfrak{g})$ and the PBW Theorem

Recall that a collection of elements of a Lie algebra are said to be generators of the Lie algebra if the smallest subalgebra containing them is the Lie algebra itself. Then we can define a special Lie algebra with applications in physics.

The **Heisenberg algebra** $\mathcal{H}$ is the Lie algebra of dimension $2n+1$ that is algebraically generated by the generators X_i, Y_j $(i, j = 1, 2, \ldots n)$ and Z which are subject to the following Lie bracket relations

$$[X_i, Y_j] = C\delta_{ij}, \quad [X_i, Z] = 0, \quad [Y_j, Z] = 0,$$

so that Z is a central element. We also have that $\mathcal{H}$ is an associative Lie algebra.

Example 3.1. We can construct a basis for $\mathfrak{U}(\mathcal{H})$, where $\mathcal{H}$ is the 3-dimensional case. We will begin by considering the bracket structure of $\mathcal{H}$, which is $[X, Y] = Z,\ [X, Z] = 0 = [Y, Z]$. Thus, for $\mathfrak{U}(\mathcal{H})$, we know that $XZ - ZX = 0,\ YZ - ZY = 0,$ and $XY - YX = Z$. If $\text{char}(\mathbb{F}) = 0$ then a basis for $\mathfrak{U}(\mathcal{H})$ is elements of the form $\{X^aY^bZ^c | a, b, c \in \mathbb{Z}_{\geq 0}\}$, where the basis elements are restricted to the relation $Z = XY - YX$, by the PBW Theorem. If we think of this as generators, then $\mathfrak{U}(\mathcal{H})$ is also generated by the elements $\{1, X, Y, Z\}$, where $XY - XY = Z$. This is an equivalent statement because for generating elements we are given linear combinations of elements where multiplication is allowed between the elements, as where with a basis, we are only given linear combination; therefore, for the PBW-type basis, we require the powers, but with generators of an algebra, we do not.

Definition 3.1 (Weyl Algebras)**.** A Weyl algebra, denoted A_n, is a ring of differential operators with polynomial coefficients (in n variables),

$$f_n(X_i)\partial^n_{X_i} + f_{n-1}(X_i)\partial^{n-1}_{X_i} + \cdots + f_1(X_i)\partial_{X_i} + f_0(X_i).$$

This algebra is generated by X_i and ∂_{X_i}.

Remark 2. If $\mathbb{F}$ is a field and $\mathbb{F}[X]$ is the ring of polynomials in one variable X with coefficients from $\mathbb{F}$, then each f_i lives in $\mathbb{F}[X]$ where ∂_X is the derivative with respect to X.

We can generalize for the n^{th} dimensional Heisenberg and say that $\mathfrak{U}(\mathcal{H}) \cong A_n$. This leads to applications in physics because the Weyl algebra "is isomorphic to the algebra of operators polynomials in the position and momenta (i.e textbook quantum mechanics) of which only the associative algebra structure is retained..." [Bek, 2005]. The applications of $\mathcal{H}$ are rooted in Werner Heisenberg's *uncertainty principle*. This "uncertain" relation corresponds to the position and momentum of subatomic particles such as protons, neutrons, and electrons. Heisenberg constructed the algebra $\mathcal{H}$ to study this quantum movement via "matrix mechanics". However, this way of viewing quantum mechanics was slow to develop. Around the same time as Heisenberg, Erwin Schrödinger was studying wave mechanics, which he formulated in the wave equation named in his honor. It was later found, by Schrödinger, that the Heisenberg's algebra was equivalent to the wave equations under a certain transformation.

This equivalency allowed physicists to use two very familiar tools to study the behavior of waves. When Louis de Broglie, who discovered the theoretical existence of matter waves, published his results he was able to give a more powerful application to physics. His result allows physicists to use matrix mechanics to also model light and matter. By using the universal enveloping algebra of $\mathcal{H}$, the structure becomes even more familiar, namely a polynomial algebra with the restriction discussed previously. More information on the history and development of this topic can be found at [Cas, 2013].

4 Conclusion

In summary we have shown that for every Lie algebra, $\mathfrak{g}$ there exists a unique associative algebra (with unit), denoted $\mathfrak{U}(\mathfrak{g})$, which inherits the relationship $[a, b] = ab - ba$ so that $\mathfrak{U}(\mathfrak{g})$ can be viewed an a larger algebra which "envelops" the Lie algebra. Moreover, we have shown that there is a linear injective mapping from the $\mathfrak{U}(\mathfrak{g})$ to the Lie algebra. From this existence, we observed that $\mathfrak{U}(\mathfrak{g})$ can be viewed as the maximal associative algebra over an arbitrary field with unity generated by $\mathfrak{g}$. We then stated a variation of the PBW Theorem which gives us a basis for $\mathfrak{U}(\mathfrak{g})$. We then showed that for any abelian Lie algebra, the corresponding universal enveloping algebra is isomorphic to a polynomial algebra. We further discussed what $\mathfrak{U}(\mathfrak{sl}(3, \mathbb{F}))$ looks like as a PBW-type basis and concluded with some applications to physics, namely the Heisenberg Lie algebra.

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