On Crystal Basis Theory of $\mathfrak{sl}_2$

Ethan A. Smith

Appalachian State University
MAA-SE Conference

Friday, March 14, 2014

Lie Algebras

Definition

Let $\mathfrak{g}$ be a vector space over a field $\mathbb{F}$, with an operation $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, denoted $(x, y) \mapsto [x, y]$ and called the *Lie bracket*; $\mathfrak{g}$ is called a **Lie algebra** over $\mathbb{F}$ if the following axioms are satisfied:

(L1) $[\alpha x + \beta y, z] = \alpha[x, z] + \beta[y, z]$,

(L2) $[x, \alpha y + \beta z] = \alpha[x, y] + \beta[x, z]$,

(L3) $[x, x] = 0$ for all x in $\mathfrak{g} \Rightarrow [x, y] = -[y, x]$,

(L4) $[x, [y, x]] + [y, [z, x]] + [z, [x, y]] = 0$, where $\alpha, \beta \in \mathbb{F}, x, y, z \in \mathfrak{g}$.

Cross-Product Lie Algebra

Let $\mathfrak{g} = \mathbb{R}^3$, and $\{i, j, k\}$ are the usual unit vectors along the coordinate axes. We know $\{i, j, k\}$ for a basis of $\mathfrak{g}$, then we define the bracket structure to be

$$[i, j] = k, \quad [j, k] = i, \quad [i, k] = -j.$$

A pictorial representation would be:

$$\begin{tikzcd} & i \arrow[dr, bend left] & \\ k \arrow[ur, bend left] & & j \arrow[ll, bend left] \end{tikzcd}$$

Let's check the Jacobi identity to see if this is a Lie algebra.

$$[i, [j, k]] + [j, [k, i]] + [k, [j, i]] = [i, i] + [j, j] + [k, -k] = 0$$

General and Special Linear Lie Algebra I

Example

Consider all $n \times n$ matrices. Then if we define $[A, B] : \ \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ where $[A, B] = AB - BA$ is the commutator bracket, then this is a Lie algebra, denoted $\mathfrak{gl}_n$.

Example

Consider all $n \times n$ matrices whose trace is zero. Then this is also a Lie subalgebra of $\mathfrak{gl}_n$, denoted $\mathfrak{sl}_n$.

General and Special Linear Lie Algebra II

Example

Consider the matrices

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \ f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \ h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

These matrices form a basis for $\mathfrak{sl}_2$ with Lie bracket structure defined to be

$$[h, e] = 2e, \ [h, f] = -2f, \ [e, f] = h.$$

This will be the driving example of this talk.

Representation Theory

Definition

A vector space V is called an $\mathfrak{g}$-**module** if there is a mapping $\mathfrak{g} \times V \to V$, denoted by $(x, v) \mapsto x \cdot v$, which satisfies the following relationships:

(M1) $(\alpha x + \beta y) \cdot v = \alpha(x \cdot v) + \beta(y \cdot v)$,

(M2) $x \cdot (\alpha v + \beta w) = \alpha(x \cdot v) = \beta(x \cdot w)$,

(M3) $[x, y] \cdot v = x \cdot (y \cdot v) - y \cdot (x \cdot v)$, where $x, y \in \mathfrak{g}$, $v, w \in V$, and $\alpha, \beta \in \mathbb{F}$.

Representation of $\mathfrak{sl}_2$

Let V be a finite dimensional irreducible $\mathfrak{sl}_2$-module. V decomposes into a direct sum of eigenspaces for h:

$$V = \bigoplus_{\lambda \in \mathbb{F}} V_\lambda, \text{ where } V_\lambda = \{v \in V | \ h \cdot v = \lambda v\}.$$

Vector Representation I

Again, recall the basis for $\mathfrak{sl}_2$. The $\mathfrak{g}$-module is $\mathbb{C}$ and the action is (left) matrix multiplication.

$$\mathrm{span}\left\{ e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\ f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},\ h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\}.$$

The "highest weight vector" is $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, with weight 1. We wish to consider the "action" of f on the vector.

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix},\quad \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Vector Representation II

So, if $v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $v_{-1} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, we have the following picture

$$\begin{tikzcd} \times & v_1 \arrow[l, bend left, "e"] \arrow[r, "f"] & v_{-1} \arrow[l, bend left, "e"] \arrow[r, "f"] & \times \end{tikzcd}$$

Action of e, f, and h

$$\begin{tikzcd} V_\lambda \arrow[loop above, "h"] \arrow[r, "f"] & V_{\lambda-2} \arrow[l, bend left, "e"] \arrow[loop above, "h"] \arrow[r, "f"] & \cdots \arrow[l, bend left, "e"] \arrow[r, "f"] & V_{-\lambda+2} \arrow[l, bend left, "e"] \arrow[loop above, "h"] \arrow[r, "f"] & V_{-\lambda} \arrow[l, bend left, "e"] \arrow[loop above, "h"] \end{tikzcd}$$

Tensor Product

Definition

Let V and W be vector spaces of a field $\mathbb{F}$. We construct the free vectors space $S = \mathbb{F}(V \times W)$. Now we construct a subspace of S, call it R, generated by the following relations:

$$R = \begin{cases} (\alpha v_1 + \beta v_2, w) - [\alpha(v_1, w) + \beta(v_2, w)], \\ (v, \alpha w_1 + \beta w_2) - [\alpha(v, w_1) + \beta(v, w_2)], \end{cases}$$

where $v_1, v_2 \in V$; $w_1, w_2 \in W$; $\alpha, \beta \in \mathbb{F}$. Then the S/R is the tensor product, namely $(V, W) + R \equiv V \otimes W$.

Definition of UEA

We can think of the UEA as an associative algebra with the Lie algebra structure defined on it.

$$i : \mathfrak{g} \xrightarrow{\text{linear}} \mathfrak{U}(\mathfrak{g}), \;\; i([x,y]) = i(x)i(y) - i(y)i(x), \text{ for all } x, y \in \mathfrak{g}.$$

$$\begin{tikzcd}
\mathfrak{g} \arrow[r, "j"] \arrow[dr, "i"] & A \\
 & \mathfrak{U}(\mathfrak{g}) \arrow[u, "\exists!\ \theta"]
\end{tikzcd}$$

Using the UEA I

Example

Let V be a highest weight $\mathfrak{U}(\mathfrak{sl}_2)$-module with highest weight vector v_i with weight i. Consider $V \otimes V$. An obvious and natural basis would be

$$v_1 \otimes v_1, \quad v_1 \otimes v_{-1}, \quad v_{-1} \otimes v_1, \quad v_{-1} \otimes v_{-1}.$$

We define an action on a tensor (extended linearly) to be as follows:

$$x \cdot (a \otimes b) = (x \cdot a) \otimes b + a \otimes (x \cdot b).$$

Then we wish to consider f acting on the highest weight, $v_1 \otimes v_1$.

Using the UEA II

$$\begin{tikzcd} \times & v_1 \arrow[l, bend left, "e"] \arrow[r, "f"] & v_{-1} \arrow[l, bend left, "e"] \arrow[r, "f"] & \times \end{tikzcd}$$

$$\begin{tikzcd} v_1 \otimes v_1 \arrow[d, "f"] \\ v_{-1} \otimes v_1 + v_1 \otimes v_{-1} \arrow[d, "f"] \\ v_{-1} \otimes v_{-1} + v_{-1} \otimes v_{-1} \end{tikzcd}$$

Quantum Groups

What is a Quantum Group?

We like to think of the Quantum Group as a "deformation" of $\mathfrak{U}(\mathfrak{g})$. So, quantum groups are not groups, they are a non-commutative, associative algebra over the field $\mathbb{F}(q)$.

Why Develop Crystal Basis Theory? I

Remember this picture? He isn't in $\mathfrak{U}_q(\mathfrak{sl}_2)$.

$$\begin{array}{c} v_1 \otimes v_1 \\ \Big\downarrow f \\ v_{-1} \otimes v_1 + v_1 \otimes v_{-1} \\ \Big\downarrow f \\ v_{-1} \otimes v_{-1} + v_{-1} \otimes v_{-1} \end{array}$$

Why Develop Crystal Basis Theory? II

Example (Cont.)

So, the actual basis for this tensor product in $\mathfrak{U}_q(\mathfrak{sl}_2)$ is

$$V(2) = \begin{cases} v_1 \otimes v_1, \\ v_{-1} \otimes v_1 + q v_1 \otimes v_{-1}, \\ v_{-1} \otimes v_{-1}, \end{cases}$$

$$V(0) = \begin{cases} v_{-1} \otimes v_1 - q v_1 \otimes v_{-1}. \end{cases}$$

If $q = 1$, we have the the elements in $\mathfrak{U}(\mathfrak{sl}_2)$. But, it'd be nice if $q = 0$, right?

Crystal Graphs

Let $V = \mathfrak{U}_q(\mathfrak{sl}_3)$-module, then we wish to construct the crystal graph of $V \otimes V$.

$$\begin{tikzcd} v_1 \arrow[d, "1"] \\ v_2 \arrow[d, "2"] \\ v_3 \end{tikzcd}$$

$$\begin{array}{ccc} v_1 \otimes v_1 & v_2 \otimes v_1 & v_3 \otimes v_1 \\ v_1 \otimes v_2 & v_2 \otimes v_2 & v_3 \otimes v_2 \\ v_1 \otimes v_3 & v_2 \otimes v_3 & v_3 \otimes v_3 \end{array}$$

Crystal Graphs

Let $V = \mathfrak{U}_q(\mathfrak{sl}_3)$-module, then we wish to construct the crystal graph of $V \otimes V$.

$$
\begin{tikzcd}
v_1 \arrow[d, "1"] \\
v_2 \arrow[d, "2"] \\
v_3
\end{tikzcd}
\qquad
\begin{tikzcd}
v_1 \otimes v_1 \arrow[r, "1"] & v_2 \otimes v_1 \arrow[r, "2"] \arrow[d, "1"] & v_3 \otimes v_1 \arrow[d, "1"] \\
v_1 \otimes v_2 \arrow[d, "2"] & v_2 \otimes v_2 \arrow[r, "2"] & v_3 \otimes v_2 \arrow[d, "2"] \\
v_1 \otimes v_3 \arrow[r, "1"] & v_2 \otimes v_3 & v_3 \otimes v_3
\end{tikzcd}
$$

References I

- Karin Erdmann and Mark J. Wildon, *Introduction to lie algebras*, Springer, 2006.
- Jin Hong and Seok-Jin Kang, *Introduction to quantum groups and crystal bases*, 1 ed., American Mathematical Society, 2002.
- James Humpherys, *Introduction to lie algebras and representation theory*, 8 ed., Springer-Verlag, 1997.
- Nathan Jacobson, *Lie algebra*, 1 ed., Dover Publications, 1979.
- Victor Kac, *Infinite-dimensional lie algebras*, 3 ed., Cambridge University Press, 1994.
- Masaki Kashiwara, *On crystal bases*, Canadian Mathematical Socieity Proceedings, March 1995.

References II

George Lusztig, *Finite dimenstional hopf algebras arising from quantized universal enveloping algebras*, Journal of the American Mathematical Socieity **3** (1990), 257–296.

Questions, Comments, Contact Me

Ethan A. Smith

- E-Mail: smithea3@appstate.edu
- Website: `http://smithea.weebly.com`