

On Crystal Basis Theory of \mathfrak{sl}_2

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Lie Algebras

Definition

Let \mathfrak{g} be a vector space over a field \mathbb{F} , with an operation $\mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$, denoted $(x, y) \mapsto [x, y]$ and called the *Lie bracket*; \mathfrak{g} is called a **Lie algebra** over \mathbb{F} if the following axioms are satisfied:

$$(L1) \quad [\alpha x + \beta y, z] = \alpha[x, z] + \beta[y, z],$$

$$(L2) \quad [x, \alpha y + \beta z] = \alpha[x, y] + \beta[x, z],$$

$$(L3) \quad [x, x] = 0 \text{ for all } x \text{ in } \mathfrak{g} \Rightarrow [x, y] = -[y, x],$$

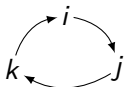
$$(L4) \quad [x, [y, x]] + [y, [z, x]] + [z, [x, y]] = 0, \text{ where } \alpha, \beta \in \mathbb{F}, x, y, z \in \mathfrak{g}.$$

Cross-Product Lie Algebra

Let $\mathfrak{g} = \mathbb{R}^3$, and $\{i, j, k\}$ are the usual unit vectors along the coordinate axes. We know $\{i, j, k\}$ for a basis of \mathfrak{g} , then we define the bracket structure to be

$$[i, j] = k, \quad [j, k] = i, \quad [i, k] = -j.$$

A pictorial representation would be:



Let's check the Jacobi identity to see if this is a Lie algebra.

$$[i, [j, k]] + [j, [k, i]] + [k, [i, j]] = [i, i] + [j, j] + [k, -k] = 0$$

General and Special Linear Lie Algebra I

Example

Consider all $n \times n$ matrices. Then if we define $[A, B] : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$ where $[A, B] = AB - BA$ is the commutator bracket, then this is a Lie algebra, denoted \mathfrak{gl}_n .

Example

Consider all $n \times n$ matrices whose trace is zero. Then this is also a Lie subalgebra of \mathfrak{gl}_n , denoted \mathfrak{sl}_n .

General and Special Linear Lie Algebra II

Example

Consider the matrices

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

These matrices form a basis for \mathfrak{sl}_2 with Lie bracket structure defined to be

$$[h, e] = 2e, \quad [h, f] = -2f, \quad [e, f] = h.$$

This will be the driving example of this talk.

Representation Theory

Definition

A vector space V is called an **\mathfrak{g} -module** if there is a mapping $\mathfrak{g} \times V \rightarrow V$, denoted by $(x, v) \mapsto x \cdot v$, which satisfies the following relationships:

$$(M1) \quad (\alpha x + \beta y) \cdot v = \alpha(x \cdot v) + \beta(y \cdot v),$$

$$(M2) \quad x \cdot (\alpha v + \beta w) = \alpha(x \cdot v) + \beta(x \cdot w),$$

$$(M3) \quad [x, y] \cdot v = x \cdot (y \cdot v) - y \cdot (x \cdot v), \text{ where } x, y \in \mathfrak{g}, v, w \in V, \\ \text{and } \alpha, \beta \in \mathbb{F}.$$

Representation of \mathfrak{sl}_2

Let V be a finite dimensional irreducible \mathfrak{sl}_2 -module. V decomposes into a direct sum of eigenspaces for h :

$$V = \bigoplus_{\lambda \in \mathbb{F}} V_\lambda, \text{ where } V_\lambda = \{v \in V \mid h \cdot v = \lambda v\}.$$

Vector Representation I

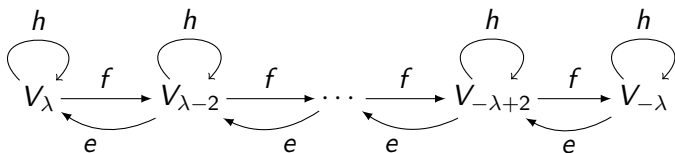
Again, recall the basis for \mathfrak{sl}_2 . The \mathfrak{g} -module is \mathbb{C} and the action is (left) matrix multiplication.

$$\text{span} \left\{ e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\}.$$

The “highest weight vector” is $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, with weight 1. We wish to consider the “action” of f on the vector.

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Action of $e, f,$ and h



Tensor Product

Definition

Let V and W be vector spaces of a field \mathbb{F} . We construct the free vectors space $S = \mathbb{F}(V \times W)$. Now we construct a subspace of S , call it R , generated by the following relations:

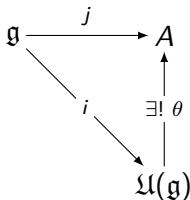
$$R = \left\{ \begin{array}{l} (\alpha v_1 + \beta v_2, w) - [\alpha(v_1, w) + \beta(v_2, w)], \\ (v, \alpha w_1 + \beta w_2) - [\alpha(v, w_1) + \beta(v, w_2)], \end{array} \right.$$

where $v_1, v_2 \in V$; $w_1, w_2 \in W$; $\alpha, \beta \in \mathbb{F}$. Then the S/R is the tensor product, namely $(V, W) + R \equiv V \otimes W$.

Definition of UEA

We can think of the UEA as an associative algebra with the Lie algebra structure defined on it.

$$i : \mathfrak{g} \xrightarrow{\text{linear}} \mathcal{U}(\mathfrak{g}), \quad i([x, y]) = i(x)i(y) - i(y)i(x), \quad \text{for all } x, y \in \mathfrak{g}.$$



Using the UEA I

Example

Let V be a highest weight $\mathfrak{U}(\mathfrak{sl}_2)$ -module with highest weight vector v_i with weight i . Consider $V \otimes V$. An obvious and natural basis would be

$$v_1 \otimes v_1, \quad v_1 \otimes v_{-1}, \quad v_{-1} \otimes v_1, \quad v_{-1} \otimes v_{-1}.$$

We define an action on a tensor (extended linearly) to be as follows:

$$x \cdot (a \otimes b) = (x \cdot a) \otimes b + a \otimes (x \cdot b).$$

Then we wish to consider f acting on the highest weight, $v_1 \otimes v_1$.

Using the UEA II

$$\begin{array}{ccccccc}
 & & & f & & f & \\
 & & & \longrightarrow & & \longrightarrow & \\
 \times & \xleftarrow{e} & v_1 & & v_{-1} & & \times \\
 & & & \xleftarrow{e} & & &
 \end{array}$$

$$\begin{array}{c}
 v_1 \otimes v_1 \\
 \downarrow f \\
 v_{-1} \otimes v_1 + v_1 \otimes v_{-1} \\
 \downarrow f \\
 v_{-1} \otimes v_{-1} + v_{-1} \otimes v_{-1}
 \end{array}$$

Quantum Groups

What is a Quantum Group?

We like to think of the Quantum Group as a “deformation” of $\mathfrak{U}(\mathfrak{g})$. So, quantum groups are not groups, they are a non-commutative, associative algebra over the field $\mathbb{F}(q)$.

Why Develop Crystal Basis Theory? I

Remember this picture? He isn't in $\mathfrak{U}_q(\mathfrak{sl}_2)$.

$$\begin{array}{c} v_1 \otimes v_1 \\ \downarrow f \\ v_{-1} \otimes v_1 + v_1 \otimes v_{-1} \\ \downarrow f \\ v_{-1} \otimes v_{-1} + v_{-1} \otimes v_{-1} \end{array}$$

Why Develop Crystal Basis Theory? II

Example (Cont.)

So, the actual basis for this tensor product in $\mathfrak{U}_q(\mathfrak{sl}_2)$ is

$$V(2) = \begin{cases} v_1 \otimes v_1, \\ v_{-1} \otimes v_1 + qv_1 \otimes v_{-1}, \\ v_{-1} \otimes v_{-1}, \end{cases}$$

$$V(0) = \{v_{-1} \otimes v_1 - qv_1 \otimes v_{-1}.$$

If $q = 1$, we have the the elements in $\mathfrak{U}(\mathfrak{sl}_2)$. But, it'd be nice if $q = 0$, right?

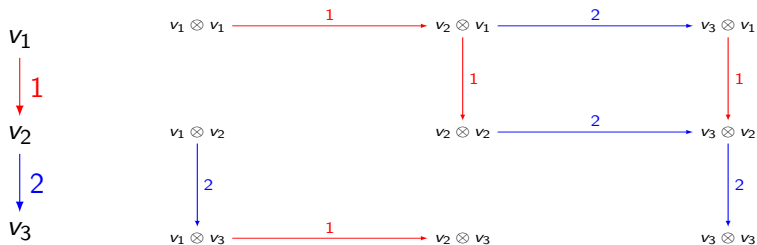
Crystal Graphs

Let $V = \mathfrak{U}_q(\mathfrak{sl}_3)$ -module, then we wish to construct the crystal graph of $V \otimes V$.









Crystal Graphs

Let $V = \mathfrak{U}_q(\mathfrak{sl}_3)$ -module, then we wish to construct the crystal graph of $V \otimes V$.



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Questions, Comments, Contact Me

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