On the Universal Enveloping Algebra: Including the Poincaré-Birkhoff-Witt Theorem

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Lie Algebra Project

February 6, 2014

Definition (Universal Enveloping Algebra)

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(iii) (Universal Property) For any associative algebra A with unit over $\mathbb{F}$ and for any linear map $j : \mathfrak{g} \to A$ satisfying $j([x, y]) = j(x)j(y) - j(y)j(x)$ for each $x, y \in \mathfrak{g}$, there exists a unique homomorphism of algebras $\theta : \mathfrak{U}(\mathfrak{g}) \to A$ such that $\theta \circ i = j$

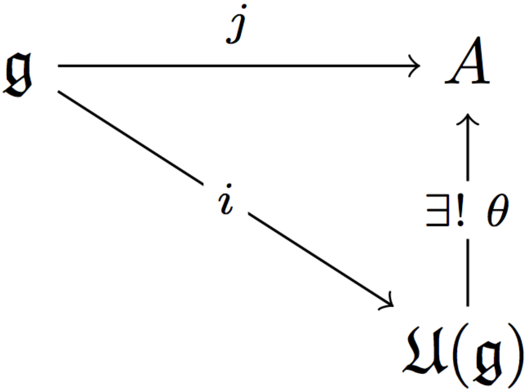

Figure: Universal Property

Properties of $\mathfrak{U}(\mathfrak{g})$.

1. For every $\mathfrak{g}$ there **exists** a **unique** $\mathfrak{U}(\mathfrak{g})$.
2. The restrictions on $\mathfrak{g}$ are none. i.e. we can have $\mathfrak{g}$ be infinite dimensional, which is great for those who study Kac- Moody Lie algebras.

Poincaré-Birkhoff-Witt Theorem

Theorem (Poincaré-Birkhoff-Witt Theorem)

(i) *The map* $i : \mathfrak{g} \to \mathfrak{U}(\mathfrak{g})$ *is injective.*

(ii) *Let* $\{x_\alpha | \alpha \in \Omega\}$ *be an ordered basis of* $\mathfrak{g}$. *Then, all the elements of the form* $x_{\alpha_1} x_{\alpha_2} \cdots x_{\alpha_n}$ *satisfying* $\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_n$ *together with 1 form a basis of* $\mathfrak{U}(\mathfrak{g})$.

We will refers to this theorem as the PBW Theorem from here on out. Any basis created from this theorem will be considered of PBW-type.

Consequences of the PBW Theorem

1. We can identity $\mathfrak{g}$ in $\mathfrak{U}(\mathfrak{g})$ from the injective mapping i.
2. We can think of $\mathfrak{U}(\mathfrak{g})$ as "enveloping" $\mathfrak{g}$.
3. We can construct a basis for $\mathfrak{U}(\mathfrak{g})$ using basis elements from $\mathfrak{g}$.

Examples of PBW Theorem

Example

Let $\mathfrak{g}$ be an abelian Lie algebra of dimension 2 with a basis $\{x_1, x_2\}$ over the field $\mathbb{F}$. Construct a basis for $\mathfrak{U}(\mathfrak{g})$ using the PBW Theorem.

Example

If $\mathfrak{g}$ is an n dimensional abelian Lie algebra, then the PBW-type basis is the polynomial algebra of n variables.

Representations of $\mathfrak{U}(\mathfrak{g})$

Theorem

A representation of $\mathfrak{g}$ can be extended naturally to a representation of $\mathfrak{U}(\mathfrak{g})$. If we let φ be a Lie algebra homomorphism and $\bar{\varphi}$ be an associative algebra homomorphism, then the following diagram commutes.