On The Argument Principle and Rouché's Theorem

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Abstract

This paper aims to discuss, in detail, the development and consequences of a famous result in Complex Analysis related to locating the roots of functions. This result is known as Rouché's Theorem. Arriving as a corollary of the Argument Principle, Rouché's Theorem gives one a tool to easily locate roots of analytic functions inside a specified closed disk (or contour). Along with examples of the Argument Principle and Rouché's Theorem, this paper also gives a proof of The Fundamental Theorem of Algebra and concludes by stating a consequence of Rouché's Theorem known as Hurwitz's Theorem, which allows one to locate the zeros of a sequence of complex functions.

Contents

1 The Argument Principle

1.1 Cauchy's Argument Principle

The first interesting theorem discussed in this paper will be the argument principle (or Cauchy's Argument Principle). This well known principle relates the difference between the number of zeros and poles of a function which is analytic on a closed piecewise-smooth curve except at finitely many points inside the curve, namely poles. In many works these functions are generally referred to as **meromorphic functions**.

Before the argument principle is introduced, we need to note that an integral of the form

$$\frac{1}{2\pi i}\int_{\gamma}\frac{f'(z)}{f(z)}dz = \frac{1}{2\pi i}\int_{\gamma} d\log f(z),$$

where $f(z)$ is analytic on a domain D, γ is a curve in D, and $f(z) \neq 0$ on γ is called the **logarithmic integral** of $f(z)$ along γ [Gamelin, 2001]. Now let's consider the following formalization of the argument principle.

Theorem 1. ***(Argument Principle)*** *[Gamelin, 2001] Let D be a bounded domain with piecewise smooth boundary ∂D, and let $f(z)$ be a meromorphic function on D that extends to be analytic on ∂D, such that $f(z) \neq 0$ on ∂D. Then*

$$\frac{1}{2\pi i}\int_{\partial D}\frac{f'(z)}{f(z)}dz = N_0 - N_\infty, \tag{1}$$

where N_0 is the number of zeros of $f(z)$ in D and N_∞ is the number of poles of $f(z)$ in D, counting multiplicities.

Proof. A proof of this theorem follows from the residue theorem. Let $f(z)$ be as hypothesized. Then $f'(z)/f(z)$ is analytic on D and its boundary save for where $f(z)$ may have a pole or a zero of order N. So, let z_0 be a zero or pole of $f(z)$, and let N be the order of $f(z)$ at z_0. Moreover, N is the order of the zero if z_0 is a zero and N is negative the order of the pole if $f(z)$ is a pole. We can define

$$f(z) = (z - z_0)^N g(z),$$

where $g(z)$ is analytic at z_0 and $g(z_0) \neq 0$. From this we can write

$$\frac{f'(z)}{f(z)} = \frac{(z-z_0)^N g'(z) + N(z-z_0)^{N-1}g(z)}{(z-z_0)^N g(z)} = \frac{g'(z)}{g(z)} + \frac{N}{z-z_0}$$

and $g'(z)/g(z)$ is analytic by definition, thus

$$\frac{1}{2\pi i}\int_{\partial D}\frac{f'(z)}{f(z)}dz = \frac{1}{2\pi i}\int_{\partial D}\left[\frac{g'(z)}{g(z)} + \frac{N}{z-z_0}\right]dz = 0 + \frac{1}{2\pi i}\int_{\partial D}\frac{N}{z-z_0}dz =$$

$$\frac{1}{2\pi i}\int_{\partial D}\frac{N}{z-z_0}dz = N.$$

Since z_0 was defined to be either a zero or a pole, by the residue theorem, if we sum over all the zeros and poles, then the sum of the residue where N_0 is the number of zeros and N_∞ is the number of poles equals

$$\sum \operatorname{Res}\left[\frac{f'(z)}{f(z)}, z_0\right] = N_0 - N_\infty.$$

Therefore, we can conclude equation (1) has been verified. □

As a result of this, the value $n = N_0 - N_\infty$ is known as the "winding number" around the origin, or the number of times the curve encircles the origin [Cain, 2001]. If n is positive, this means the complex curve winds around the origin in a counter-clockwise fashion; if n is negative, then the complex curve winds around the origin in a clockwise fashion.

Another way to state the argument principle is to think about the increase in the argument of a function around the boundary of a contour. We can define the increase in the argument of a function $f(z)$ around the boundary of D, where ∂D consists of a finite number of piecewise-smooth closed curves with positive orientation, so that D lies on the left as we traverse the curves in ∂D to be the sum of its increases around the closed curves in ∂D. The restatement is as follows:

Theorem 2. ***(Argument Principle Restated)*** *[Gamelin, 2001] Let D be a bounded domain with piecewise smooth boundary ∂D, and let $f(z)$ be a meromorphic function on D that extends to be analytic on ∂D, such that $f(z) \neq 0$ on ∂D. Then the increase in the argument of $f(z)$ around the boundary of D is 2π times the number of zeros minus the number of poles of $f(z)$ in D,*

$$\int_{\partial D} d \arg(f(z)) = 2\pi(N_0 - N_\infty) \tag{2}$$

As previously stated, we can think of the argument principle as an increase of the argument around a specified domain. This is a direct result of looking closer at the logarithm integral. We can express the logarithmic integral of the form

$$\frac{1}{2\pi i}\int_\gamma d\log f(z) = \frac{1}{2\pi i}\int_\gamma d\left[\log|f(z)| + i\arg(f(z))\right] = \frac{1}{2\pi i}\int_\gamma d\log|f(z)| + \frac{1}{2\pi}\int_\gamma d\arg(f(z))$$

From this we can see that the integral of $d\log|f(z)|$ around γ is zero since it is exact, which implies independence of path. Under the parameterization $\gamma(t) = x(t)+iy(t))$, we can conclude that

$$\frac{1}{2\pi}\int_\gamma d\arg(f(z)) = \arg f(\gamma(b)) - \arg f(\gamma(a)),$$

where the quantity $\displaystyle\int_\gamma d\arg(f(z)) = \arg f(\gamma(b)) - \arg f(\gamma(a))$ is called the increase in the argument.

1.2 Examples of Cauchy's Argument Principle

Now that we have formalized the idea of the Argument Principle, let's look at how we can use it to evaluate integrals and solve some interesting problems.

Example This examples aims to give a more "algebraic" demonstration of the argument principle. Let $f(z) = \dfrac{(z-1)^2}{(z-i)^3}$. Therefore, $f'(z) = -3(z-1)^2(z-i)^{-4} + 2(z-1)(z-i)^{-3}$ and

$$\frac{f'(z)}{f(z)} = \frac{-3(z-1)^2(z-i)^{-4} + 2(z-1)(z-i)^{-3}}{(z-1)^2 * (z-i)^{-3}} = \frac{-3}{z-i} + \frac{2}{z-1}.$$

Therefore, by the residue theorem,

$$\oint_{|z|=2} \frac{f'(z)}{f(z)} dz = \oint_{|z|=2} \left(\frac{-3}{z-i} + \frac{2}{z-1}\right) dz = 2\pi i\,(-3+2) = -2\pi i$$

One can observe that $f(z)$ has a zero of order 2 at $z = 1$ a pole of order 3 at $z = i$. So, the argument principle would determine that

$$\oint_{|z|=2} \frac{f'(z)}{f(z)} dz = 2\pi i(2-3) = -2\pi i$$

Which is exactly what we achieved. Let's consider another example.

Example Let Γ be the unit circle transposed in the counter-clockwise orientation, and let f be given by

$$f(z) = \frac{z^2+2}{z^3}.$$

How many times does the curve $f(\Gamma)$ wind around the origin?

Solution. We can observe that $f(z)$ has a pole of order 3 at the origin and zeros at $z = \sqrt{2}e^{i\pi/4}$ and $\sqrt{2}e^{-i\pi/4}$. The zeros do not lie within Γ, so only the pole of order 3 is inside Γ, therefore, $n = N_0 - N_\infty = 0 - 3 = -3$. We can conclude that the curve winds around the origin three times in a clockwise fashion.

Proposition Another interesting problem that arises from the argument principle is the following: let f be an entire function and $f(z)$ is real if and only if z is real. Show that f has at most one zero.

Proof. (*by contraction*) Suppose f has more than one root. Let R be so large that there are more than one root inside a piecewise smooth curve and the circle $|z| = R$. Since $f(R)$ and $f(-R)$ are the only real valued outputs of the function on the curve, $\{f(Re^{i\theta})|0 < \theta < \pi\}$ and $\{f(Re^{i\theta})|\pi < \theta < 2\pi\}$ lie entirely on the upper or lower half plan. Then the winding number of $f(\Gamma)$ is either 0 or 1. Thus, f can have at most one root by the Argument Principle, which is a contradiction that f has more than one root inside the circle with radius R. So, we can conclude that f has at most one root. $\square$

1.3 Final Comments on the Argument Principle

To sum up the argument principle in an informal manner, we can say that every zero of a meromorphic function f winds the image of f around the origin in an counter-clockwise fashion, as where the poles of the function winds f in a clockwise fashion around the origin. The total number of times this winding occurs is, as we have stated, the winding number, which is equivalent to the number of zeros minus the number of poles in the specified contour; moreover, it is equivalent to the increase in the argument of f around the contour (assuming the contour is closed).

In the general case of a winding number (not necessarily centered at the origin) we can calculate the winding number around γ, which we define to be piecewise smooth by

$$W(\gamma, z_0) = \frac{1}{2\pi i} \int_\gamma \frac{dz}{z - z_0} = \frac{1}{2\pi} \int_\gamma \arg(z - z_0)$$

However, for the preceding corollary, we will once again consider the origin as the point of interest for the increase in the argument.

2 Rouché's Theorem

2.1 A Corollary of the Argument Principle

Consider the following analogy, which can be found in E. B. Saff's textbook[1]. Think about trying to walk a dog on a leash in a city where lampposts and obstacle on which the leash may become entangle exist in abundance. If the dog and his walker encounter such an obstacle and the leash is long, the canine will inevitably cause the leash to become tangled. However, if the dog's walker continues to make adjustments to the length of the leash as they walk so it never quite reaches exactly to the post, then both dog and walker will wind around the post equal number of times and ultimately avoid becoming entangled in the obstacle.

Now let's extend this analogy to the complex plane. Suppose $f(z)$ is analytic on a domain D and meromorphic inside. Moreover, we can find the number of times $f(D)$ winds around the origin. If we consider "*perturbing*" $f(z)$ by some analytic function $h(z)$ to form $g(z) = f(z) + h(z)$, then we would like to know how small the perturbation h must be to guarantee that $g(D)$ winds around the origin the same number of times as $f(D)$ [Saff, 2003]. So, $h(z) = g(z) - f(z)$ become analogous to the leash, and impose the condition that the leash never extends from the dog's walker to the obstacle so that $f(D)$ and $g(D)$ have the same winding number, which can be expressed as

$$|h(z)| < |f(z)|, \quad z \in \partial C$$

The resulting formalization of this idea (coupled with argument principle) is known as Rouché's Theorem, with functions f and h having no poles inside the domain.

Theorem 3. ***(Rouché's Theorem)*** *[Gamelin, 2001] Let D be a bounded domain with piecewise smooth boundary ∂D. Let $f(z)$ and $h(z)$ be analytic on $D \cup \partial D$. If $|h(z)| < |f(z)|$ for $z \in \partial D$, then $f(z)$ and $f(z)+h(z)$ have the same number of zeros in D, counting multiplicities.*

Proof. Since $f(z)$ and $h(z)$ are analytic on ∂D, they are never zero on ∂D; moreover, their sum $f(z) + h(z)$ is never zero on the boundary. We can write $f(z) + h(z) = f(z)[1 + h(z)/f(z)]$, and obtain that

$$\arg(f(z) + h(z)) = \arg(f(z)) + \arg\left(1 + \frac{h(z)}{f(z)}\right).$$

Since $|h(z)/f(z)| < 1$, the values of $1 + h(z)/f(z)$ lie in the right half-plane. We can also say that the increase of the argument of $1+h(z)/f(z)$ around a closed boundary is zero. Therefore,

$$\arg(f(z) + h(z)) = \arg(f(z))$$

and by the restatement of the argument principle (see **Theorem 2**) we know that $f(z)$ and $f(z) + h(z)$ have equivalent winding numbers and therefore have equal number of roots in D. □

Another way of thinking about Rouché's Theorem is to consider the following statement: "In other words, the number of zeros of an analytic function remains constant under small perturbations [Gamelin, 2001]." Now we can look at some interesting examples (and theorems) that arise from this powerful tool used to located the number of roots of a function within a specified disk or contour.

[1] See page 360 [Saff, 2003]

2.2 Examples of Rouché's Theorem in Use

Example Consider the equation $e^z = 1 + 2z$ in the unit circle. We can see the apparent solution of $z = 0$, but we can also determine if there are other possible roots inside the unit circle. So, we would like to use Rouché's Theorem to count the number of zeros of $e^z - 1 - 2z$. Within the defined contour, $f(z) = -2z$ is the "big" term, so $h(z) = e^z - 1$. Based on the Taylor expansion of e^z, we estimate $h(z)$ by

$$|h(z)| = |e^z - 1| = \left| z + \frac{z^2}{2} + \frac{z^3}{6} + \frac{z^4}{24} + \cdots \right| \leq |z| + \frac{|z|^2}{2} + \frac{|z|^3}{6} + \frac{|z|^4}{24} + \cdots$$

If $|z| = 1$, the right-hand side is $e - 1 \approx 1.7 < 2 = |f(z)|$. Since $f(z)$ has only one root in the unit circle, then $f(z) + h(z)$ also only has one root. So, the equation $e^z = 1 + 2z$ has only one solution in the unit disk, namely $z = 0$.

Example We can show that if m and n are positive integers, then the polynomial

$$p(z) = 1 + z + \frac{z^2}{2} + \cdots + \cdots \frac{z^m}{m!} + 3z^n$$

has exactly n zeros in the unit circle.

Solution. If we let $p(z)$ as hypothesized, then we can consider $p(z) - 3z^n$.

$$|p(z) - 3z^n| = \left| z + \frac{z^2}{2} + \frac{z^3}{6} + \frac{z^4}{24} + \cdots \frac{z^m}{m!} \right|$$

The right-side of the equation is the m degree Maclauren polynomial of e^z, so we can write

$$|p(z) - 3z^n| = \left| 1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \frac{z^4}{24} + \cdots + \frac{z^m}{m!} \right| < e, \quad \text{for } |z| < 1$$

So, if $f(z) = 3z^n$ and $h(z) = 1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \frac{z^4}{24} + \cdots + \frac{z^m}{m!}$, then $f(z) + h(z)$ has the same number of zeros as $f(z)$, which has n roots in the unit circle.

2.3 Fundamental Theorem of Algebra

Using Rouché's Theorem, we can show a quick proof of the Fundamental Theorem of Algebra, which states that every polynomial having complex coefficients and degree greater than 1 has at least one complex root. Rouché's theorem proves an even stronger statements. Begin by letting $p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots a_1 z + a_0$. Moreover, if R is sufficiently large and $f(z) = a_n z^n$ and $h(z) = a_{n-1} z^{n-1} + \cdots a_1 z + a_0$, then for $|z| = R$, $|h(z)| < |f(z)|$. This implies that $p(z) = f(z) + h(z)$ has as many roots as $f(z)$ in $|z| < R$, which we knows has n roots.

An interesting implication of this result is the fact that n, which is equivalent to the power of the polynomial in question, is also the number of roots in R. Not only can we find the roots of the polynomial (by way of the factorization theorem for polynomials), but we can also find the value of R that gives us all the roots; thereby providing us with a sufficient domain on which the polynomial achieves all its roots. This not only verifies the Fundamental Theorem of Algebra, but a stronger consequence of what has been shown here is that every non-zero, single-variable, degree n polynomial with complex coefficients has, counted with multiplicity, exactly n roots.

Another result from this particular statement is that the field of complex numbers is the *algebraic closure*[2] for the reals. It is also important to note that although numerous proofs of the Fundamental Theorem of Algebra exist, there are no purely algebraic proofs of this long-studied result. One reason for this is due to the fact that for infinity many polynomials the roots for said polynomials are imaginary.

2.4 A Stronger Version of Rouché's Theorem

An interesting "remark" given by Irving Glicksberg in American Mathematical Monthly gives a stronger version of Rouché's Theorem.

Theorem 4. ***[Glicksberg, 1976]*** *Assume f and g are analytic within and on a rectifiable simple closed curve ∂, and that*

$$|f+g| < |f| + |g| \text{ on } \partial.$$

Then f and g have the same number of zeros within ∂.

Glicksberg states that this is a "slightly strengthened and symmetric form of the classical Rouché result ... and has almost equally simple proof" [Glicksberg, 1976]. We see from this that for well chosen complex functions f and g, if the sum of their distance is greater than the distance of their sums, then the conclusion remains the same: f and g have the same number of zeros in D (or ∂ as defined by Glicksberg). Saff also extends the dog and walker analogy to Glicksberg's statement by saying that if we let τ denote the ray extending from the "lamppost" in the direction away from the walker and if the dog is restricted to stay on one side or the other of τ — and never to cross it — then the leash will not tangle around the "lamppost" and both dog and walker will encircle the post the same number of times [Saff, 2003].

2.5 An Extension to Sequences of Functions

Although there are numerous results from Rouché's Theorem one useful theorem used to analyze the behavior of zeros of convergent sequence, which is a direct result of Rouché's theorem, is known as Hurwitz's Theorem.

Theorem 5. ***(Hurwitz's Theorem)***[3] *[Cain, 2001] Suppose U is an open set and suppose that a sequence $\{f_n(z)\}$ of functions analytic on U converges uniformly to the function $f(z)$. Suppose further that f is not zero on the disk $D(z_0, R) \subset U$. Then there exists an $n^* \in \mathbb{N}$ such that for all $n \geq N$, the sequence of functions $\{f_n(z)\}$ and $f(z)$ have the same number of zeros inside $D(z_0, R)$.*

Proof. [Ponnusamy, 2006] Begin by letting $\{f_n(z)\}$ be as hypothesized. Since $\{f_n(z)\}$ is a sequence of functions that converge uniformly to $f(z)$, the $f(z)$ is also analytic and non-zero in $D(z_0, R)$. Let $m > 0$ denote the minimum of $|f(z)|$ on $D(z_0, R)$. By uniform convergence of $\{f_n(z)\}$ in $D(z_0, R)$, we have that for $n > N(m)$ that

$$|f_n(z) - f(z)| < m \leq |f(z)|, \text{ in } D(z_0, R).$$

By Rouché's theorem, the number of zeros of $f(z)$ inside $D(z_0, R)$ is equivalent to the number of zeros of

$$f(z) + (f_n(z) - f(z)) = f_n(z) \text{ for } n > N.$$

□

[2][Dummit, 2004]"A field K is said to be *algebraically closed* if every polynomial with coefficients in K has a root in K of the field of real numbers."

[3]Some changes have been made to the wording of this theorem.

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