ON QUANTUM GROUPS AND CRYSTAL BASES

A Directed Research
by
ETHAN A. SMITH

Submitted to the Department of Mathematical Sciences
of Appalachian State University
in Partial Fulfillment of the Requirements
for the Degree of Master of Arts

May 2014
Department of Mathematical Sciences

A B S T R A C T

ON QUANTUM GROUPS AND CRYSTAL BASES

Ethan A. Smith
B.S, North Carolina Central University
M.A, Appalachian State University

Adviser: Vicky W. Klima

With near a century of development beginning with Sophius Lie in his 1871 Ph.D thesis *On a class of geometric transformations*, Lie algebras, and their corresponding groups, have become an essential tool for many mathematicians and physicists. Their beautiful behaviors and representation theory, despite being fairly new historically speaking, are still studied and developed from an abstract perspective. The purpose of the research presented herein is two fold in nature: the first is an expository look at the construction of quantum groups and crystal bases of Lie algebras of the finite and affine type, the second is an investigative search for combinatorial relationships of affine Kac-Moody algebras highest weight multiplicities via crystal basis theory. A particular focus on the affine Kac-Moody algebra know as the quantum special linear Lie algebra, denoted $\widehat{\mathfrak{sl}}_2$, is heavily considered in the later chapters.

ACKNOWLEDGEMENTS

I would like to begin by thanking my adviser Dr. Vicky Klima for her incredible patience and support during the completion of this research. If not for her guidance, this would not have been possible, and for that I am grateful. I am also thankful of my family for their support throughout my matriculation at both the undergraduate and graduate level. If not for their love, emotional and financial support, I would not be where I am today. Moreover, I thank the Department of Mathematical Sciences at Appalachian State University for providing me with the resources to achieve this goal.

DEDICATIONS

I dedicate this work to Bobby G. Lowe and Helen B. Smith, two of my loving grandparents who were unable to see me reach this level of success. Although they are no longer with me to celebrate this achievement that they passionately encouraged, I know that I have honored their legacy and made them proud.

Contents

List of Figures

Chapter 1

Review of Lie Algebras

1.1 Very Brief Historical Exposition of Lie Algebras

The history of the development of Lie groups, algebras, and representations thereof is an immense and growing subject. Dating back to Sophus Lie (1842-1899), Lie algebras have been a topic of discussion for over 100 years. Lie's investigation began with all possible local group actions on manifolds. The Lie algebra is the simplest example where the local group action acts on itself by left (or right) translation. By axiomatic construction, the Lie algebra of a Lie group is a linear object. Wilhelm Killing (1847-1923) proposed that prior to classifying all group actions the classifications of Lie algebras should be completely developed. The classification of all finite simple Lie algebras was completed by Élie Cartan (1869-1951) by building upon the work of Lie, Killing, and Friedrick Engel (1861-1941).

1.2 Basics of Lie Algebras

Throughout this paper, unless otherwise noted, $\mathbb{F}$ will denote an algebraically closed field of characteristic zero. To begin our discussion of Lie algebras, we will define a Lie algebra and the axioms that govern them.

Definition 1.2.1. Let $\mathfrak{g}$ be a vector space over a field $\mathbb{F}$, with an operation $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, denoted $(x, y) \mapsto [x, y]$ and called the *bracket* or *commutator* of x and y, then $\mathfrak{g}$ is called a **Lie algebra** over $\mathbb{F}$ if for all $x,\ y,\ z \in \mathfrak{g}$ and $\alpha,\ \beta \in \mathbb{F}$ the following axioms are satisfied:

(L1) $[\alpha x + \beta y, z] = \alpha[x, z] + \beta[y, z]$,

(L2) $[x, \alpha y + \beta z] = \alpha[x, y] + \beta[x, z]$,

(L3) $[x, x] = 0$,

(L4) $[x, [y, x]] + [y, [z, x]] + [z, [x, y]] = 0$.

Since axiom (L4) is such a defining relationship for a Lie algebra, we give it a name: the *Jacobi identity*. Note that axioms (L1) and (L2) ensure that the bracket is bilinear. Moreover, bilinearity of the bracket and (L3) implies that $[x, y] = -[y, x]$. This can easily be verified by letting $[x + y, x + y] = 0$ and simplifying the equation. If $\mathfrak{g}$ is a Lie algebra such that for every x and y in $\mathfrak{g}$ the bracket is zero, then we call this an **abelian** Lie algebra.

Example 1.2.1. Let us look at a familiar three-dimensional Lie algebra. Let $\mathfrak{g} = \mathbb{R}^3$ be the Euclidean 3-space and $\{i, j, k\}$ be the unit vectors along the coordinate axes. Then $\{i, j, k\}$ is a basis for $\mathfrak{g}$ over $\mathbb{R}$ with the bracket structure on $\mathfrak{g}$ begin the vector cross-product, namely:

$$[i, j] = k, \quad [j, k] = i, \quad [i, k] = -j.$$

We will force (L1)-(L3) to hold by extending the bracket structure linearly to $\mathfrak{g}$ and noting that the cross-product of a vector with itself is zero, thus $\forall x \in \mathfrak{g}$ $[x, x] = 0$. Finally, we need to check that the Jacobi identity holds for the defined bracket structure. So, we write

$$[i, [j, k]] + [j, [k, i]] + [k, [j, i]] = [i, i] + [j, j] + [k, -k] = 0$$

and show that (L4) holds. Thus this is a Lie algebra called the *cross-product Lie algebra*.

Definition 1.2.2 (Lie subalgebra)**.** We say that a Lie algebra, K, is a (Lie) **subalgebra** of $\mathfrak{g}$ if the following are satisfied:

(1) K is a vector subspace of $\mathfrak{g}$.

(2) If x and y are in K, then $[x, y]$ is also in K.

Now we can look at a Lie algebra and ones of its subalgebras that will be of great importance to us.

Definition 1.2.3. An **associative algebra** A is a vector space V over a field $\mathbb{F}$ which contains an associative, bilinear vector product $\cdot : V \times V \to V$. If there is some element $1 \in V$ such that $1 \cdot a = 1 = a \cdot 1$ for every $a \in A$, then A is unital, or "has a unit".

Remark 1. If we define $[-, -] : A \times A \to A$ to be the commutator bracket ($[x, y] = x \cdot y - y \cdot x$) where '$\cdot$' denotes the associative product in A, then if we endow A with the commutator brackt it becomes a Lie algebra. This beautiful generalization for associative algebras give way to the next example.

Example 1.2.2. Consider all $n \times n$ matrices over $\mathbb{F}$, which is an associative algebra and denoted $\mathfrak{gl}(n, \mathbb{F})$ or $\mathfrak{gl}_n$ for short. We can define the bracket on $\mathfrak{gl}_n$ to be $[-, -] : \mathfrak{gl}_n \times \mathfrak{gl}_n \to \mathfrak{gl}_n$ such that $[A, B] = A \cdot B - B \cdot A$ where $A, B \in \mathfrak{gl}_n$. With the bracket defined in this way $\mathfrak{gl}_n$ becomes a Lie algebra which we call the *general linear Lie algebra*. Now let $\mathfrak{sl}_n(\mathbb{F})$ be all $n \times n$ matrices whose trace is zero. We claim that this is a Lie subalgebra of $\mathfrak{gl}_n$. To prove this we must show that properties (1) and (2) hold from Definition 1.2.2.

Let A and B be matrices from $\mathfrak{sl}_n$. To prove $\mathfrak{sl}_n$ is in fact a subspace we show that an element of the form $\alpha A + \beta B$ has a trace of zero. So, we write $\text{trace}(\alpha A + \beta B) = \alpha\text{trace}(A) + \beta\text{trace}(B) = 0$ since $A, B \in \mathfrak{sl}_n$. So, $\mathfrak{sl}_n$ is a vector subspace. Now we must show that $\mathfrak{sl}_n$ is closed under the bracket. To do this we will show that for any A and B in $\mathfrak{sl}_n$, $[A, B]$ is also in $\mathfrak{sl}_n$; to do this we show that the trace of $[A, B]$ is zero. This fact follows from the linearity of the trace operator and the fact that trace does not depend upon the ordering of the matrix product, i.e $\text{trace}(AB) = \text{trace}(BA)$. Thus $\text{trace}([A, B]) = \text{trace}(AB - BA) = \text{trace}(AB) - \text{trace}(BA) = 0$.

We have shown that $\mathfrak{sl}_n$ is a vector subspace that is closed under the commutator bracket so we can say that $\mathfrak{sl}_n$ is a Lie subalgebra of $\mathfrak{gl}_n$. However, despite being a subalgebra, $\mathfrak{sl}_n$ is a Lie algebra in its own right.

Chapter 2

Representation Theory of $\mathfrak{sl}_n$

2.1 Basic Definitions

Let $\mathfrak{g}$ be a Lie algebra and V be a vector space over $\mathbb{F}$.

Definition 2.1.1. A **representation** of $\mathfrak{g}$ on V is a Lie algebra homomorphism $\varphi : \mathfrak{g} \to \mathfrak{gl}(V)$.

Definition 2.1.2. A vector space V is called an $\mathfrak{g}$**-module** if for all $x, y \in \mathfrak{g}$, $v, w \in V$, and $\alpha, \beta \in \mathbb{F}$ there is a mapping $\mathfrak{g} \times V \to V$, denoted by $(x, v) \mapsto x \cdot v$, which satisfies the following relationships:

(M1) $(\alpha x + \beta y) \cdot v = \alpha(x \cdot v) + \beta(y \cdot v)$,

(M2) $x \cdot (\alpha v + \beta w) = \alpha(x \cdot v) = \beta(x \cdot w)$,

(M3) $[x, y] \cdot v = x \cdot (y \cdot v) - y \cdot (x \cdot v)$.

In light of these definitions we can observe that a representation $\varphi : \mathfrak{g} \to \mathfrak{gl}(V)$ of a Lie algebra on a vector space V defines an $\mathfrak{g}$-module structure on V by the equality $x \cdot v = \varphi(x)(v)$ for any $x \in \mathfrak{g}$ and $v \in V$. Moreover, if V is a $\mathfrak{g}$-module, we can define a representation by the same equation. Therefore, if we are given a representation of $\mathfrak{g}$, then we can define a $\mathfrak{g}$-module; if we are given a $\mathfrak{g}$-module, then we can define a representation of $\mathfrak{g}$.

Let V and W be $\mathfrak{g}$ modules, then we can define a **homomorphism of** $\mathfrak{g}$**-modules** by a linear map $\phi : V \to W$ where $\phi(x \cdot v) = x \cdot \phi(v)$. In the event that ϕ is an isomorphism of vector spaces, then ϕ is an isomorphism of $\mathfrak{g}$-modules. When this happens, the two modules are said to have **equivalent** representations.

Let W be a subspace of a $\mathfrak{g}$-module V, then W is called a **submodule** of V if

$$x \cdot W \subset W \text{ for all } x \in \mathfrak{g}.$$

Let W be a submodule of a vector space V, then the quotient V/W becomes an $\mathfrak{g}$-module with the action of $\mathfrak{g}$ defined to be

$$x \cdot (v + W) = (x \cdot v) + W \text{ for } x \in \mathfrak{g},\ v \in V.$$

If the only submodules of V are zero and itself, then V is said to be **irreducible**.

2.2 The special linear Lie algebra

2.2.1 Representation of $\mathfrak{sl}_2(\mathbb{F})$

Again we will consider the special linear Lie algebra when $n = 2$. Let $\mathfrak{g} = \mathfrak{sl}_2$ over $\mathbb{F}$, then a basis for $\mathfrak{g}$ consists of three matrices, namely

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

With this basis, the bracket structure of $\mathfrak{g}$ is $[h, e] = 2e,\ [h, f] = 2f,\ [e, f] = h$. It is interesting to note that the assumption of $\mathbb{F}$ being an algebraically closed field is to ensure all the necessary eigenvalues exist in $\mathbb{F}$. Consider the mapping $\text{ad}_x(y) = [x, y]$, for $x, y \in \mathfrak{g}$. This is a Lie algebra homomorphism, which can be proven by showing that $\text{ad}_x([y, z]) = [ad_x(y), ad_x(z)]$, which is a result of the Jacobian identity. Now let V be a finite dimensional irreducible $\mathfrak{sl}_2$-module. V decomposes into a direct sum of eigenspaces for ad_h:

$$V = \bigoplus_{\lambda \in \mathbb{F}} V_\lambda, \text{ where } V_\lambda = \{v \in V \mid h \cdot v = \lambda v\}. \tag{2.1}$$

If $V_\lambda \neq 0$ for some $\lambda \in \mathbb{F}$, then we call λ a **weight** of V, V_λ is the corresponding λ-**weight space**, and the dimension of V_λ is called the **weight multiplicity** of λ.

The next two lemmas are suggested as an exercise problem in [HK02, Ch1], but important to realize the behavior of the representation, so we provide a proof here.

Lemma 2.2.1. *If $v \in V_\lambda$, then the following are true:*

(i) $e \cdot V_\lambda \subset V_{\lambda+2}$,

(ii) $f \cdot V_\lambda \subset V_{\lambda-2}$.

Proof. Let v be an element from V_λ.

(i) We wish to show that $e \cdot v \in V_{\lambda+2}$. Consider $h \cdot (e \cdot v)$, then we can write

$$\begin{aligned} h \cdot (e \cdot v) &= [h, e] \cdot v + e \cdot (h \cdot v) && \text{Definition 2.1.2, (3)} \\ &= 2e \cdot v + \lambda e \cdot v \\ &= (2 + \lambda) e \cdot v \end{aligned}$$

(ii) Now we wish to show that $f \cdot v \in V_{\lambda-2}$. Consider $h \cdot (f \cdot v)$, then we can write

$$\begin{aligned} h \cdot (f \cdot v) &= [h, f] \cdot v + f \cdot (h \cdot v) && \text{Definition 2.1.2, (3)} \\ &= -2f \cdot v + \lambda f \cdot v \\ &= (-2 + \lambda) f \cdot v \end{aligned}$$

□

An immediate result of this lemma is the realization of how $e,\ f$, and h act on the weight spaces. From this we see that the action of f moves us down in weight space, e moves us up in weight space, and h "circles" us around the weight space. We can illustrate this in the following figure.

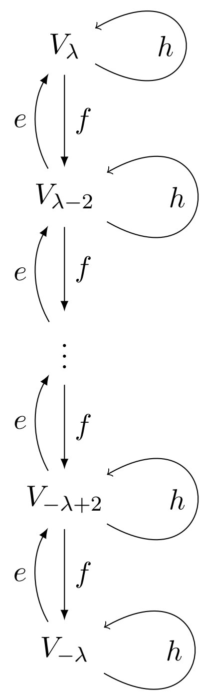


Figure 2.1

Lemma 2.2.2. *Let V be a finite dimensional irreducible $\mathfrak{sl}_2$-module over $\mathbb{F}$, then*

(i) $f \cdot f^{(i)} v_0 = (i+1) f^{(i+1)} v_0$,

(ii) $h \cdot f^{(i)} v_0 = (\lambda - 2i) f^{(i)} v_0$,

(iii) $e \cdot f^{(i)} v_0 = (\lambda - i + 1) f^{(i-1)} v_0$,

where $v_{-1} = 0 = v_{m+1}$ *for* $i = 0, \ldots, m$ *and* $f^{(i)} = \frac{f^i}{i!}$.

Proof. Since the dimension of V is finite, there exists a weight λ such that $\lambda + 2$ is not a weight. Choose a vector $0 \neq v_0 \in V_\lambda$. Now define the vectors v_i for $i = 0, \ldots, m$ by

$$v_i = \frac{1}{i} f \cdot v_{i-1}. \tag{2.2}$$

By Equation 2.2 we can express v_i in the following fashion

$$v_i = \frac{1}{i} f \cdot v_{i-1} - \frac{1}{i(i-1)} f^2 \cdot v_{i-2} = \frac{1}{i(i-1)(i-2)} f^3 \cdot v_{i-3} = \cdots = \frac{1}{i!} f^i v_0 = f^{(i)} \cdot v_0. \tag{2.3}$$

Here $v_i \in V_{\lambda-2i}$ by Lemma 2.2.1. So, we have

(i) This follows directly from the (2.2).

(ii) Given the fact that $v_i = \frac{1}{i!}f^i v_0$ from Equation (2.3), we know that $v_i \in V_{\lambda-2i}$, thus

$$h \cdot v_i = (\lambda - 2i)v_i = (\lambda - 2i)\frac{1}{i!}f^i v_0 = (\lambda - 2i)f^{(i)}v_0.$$

(iii) (By induction) Recall the relationship $[e, f] = e \cdot f - f \cdot e = h$ and the fact that we assumed $V_{\lambda+2} = \{0\}$ such that $e \cdot v_0 = 0$. Then we begin by assuming that $e \cdot f^{(i)} \cdot v_0 = (\lambda - i + 1)f^{(i-1)} \cdot v_0$. So we write

$$\begin{aligned}
(i+1)e \cdot f^{(i+1)} \cdot v_0 &= e \cdot (i+1)f^{(i+1)}v_0 \quad \text{(by (2.2))} \\
&= e \cdot (f \cdot f^{(i)}(v_0) \quad \text{(by (i))} \\
&= ([e, f] + f \cdot e) \cdot f^{(i)}v_0 \\
&= (h \cdot + f \cdot e) \cdot f^{(i)}v_0 \\
&= h \cdot f^{(i)}v_0 + f \cdot e \cdot f^{(i)}v_0 \\
&= (\lambda - 2i)f^{(i)}v_0 + f \cdot (\lambda - i + 1)f^{(i-1)}v_0 \quad \text{(by induction and (ii))} \\
&= (\lambda - 2i)f^{(i)}v_0 + (\lambda - i + 1)f \cdot f^{(i-1)}v_0 \\
&= (\lambda - 2i)f^{(i)}v_0 + i(\lambda - i + 1)f^{(i)}v_0 \\
&= (\lambda - i + i\lambda - i^2)f^{(i)}v_0 \\
&= (i+1)(\lambda - i)f^{(i)}v_0
\end{aligned}$$

Therefore, by induction $e \cdot f^{(i)}v_0 = (\lambda - i + 1)f^{(i-1)}v_0$.

□

2.2.2 Representation of $\mathfrak{sl}_n(\mathbb{F})$

Now that we have looked at the three dimensional special linear Lie algebra, we can consider the general special linear Lie algebra, denoted $\mathfrak{sl}_n(\mathbb{F})$. We define this Lie algebra to be all $n \times n$ with a trace of zero. The matrices of the form

$$E_{ij} \quad (i \neq j), \qquad E_{i,i} - E_{i+1,i+1} \quad (i = 1, \ldots, n-1)$$

form a basis of $\mathfrak{sl}_n(\mathbb{F})$. Here E_{ij} denotes the $n \times n$ elementary matrix whose (i, j)-entry is one and every other entry is zero. Conforming to traditional notational convention of e, f, and h we define these elements to be of the form $e_i = E_{i,i+1}$, $f_i = E_{i+1,i}, h_i = E_{i,i} - E_{i+1,i+1} \quad (i = 1, \ldots, n-1)$ satisfying the relations

$$[e_i, f_j] = \delta_{ij}h_i, \tag{2.4}$$

$$[h_i, e_j] = \begin{cases} 2e_j & \text{if } i = j, \\ -e_j & \text{if } |i-j| = 1, \\ 0 & \text{if } |i-j| > 1., \end{cases} \tag{2.5}$$

$$[h_i, f_j] = \begin{cases} -2f_j & \text{if } i = j, \\ f_j & \text{if } |i-j| = 1, \\ 0 & \text{if } |i-j| > 1. \end{cases} \tag{2.6}$$

Chapter 3

Universal Enveloping Algebra

The universal enveloping algebra, denoted $\mathfrak{U}$, is an essential tool for studying representations and more generally for studying homomorphisms of a Lie algebra $\mathfrak{g}$ into an associative algebra with an identity element [Jac79]. Our motivation for constructing $\mathfrak{U}$ is as follows: we wish to view $\mathfrak{g}$ as an associative algebra, namely $\mathfrak{U}$, via the representations of $\mathfrak{U}$. This result is obtained from an important property of $\mathfrak{U}$ which states that $\mathfrak{g}$ is isomorphic to a subalgebra of $\mathfrak{U}(\mathfrak{g})$. From this isomorphism we get a faithful representation for every $\mathfrak{g}$.

3.1 Universal Enveloping Algebra

Definition 3.1.1. Let $\mathfrak{g}$ be a Lie algebra over an arbitrary field $\mathbb{F}$. The **universal enveloping algebra** of $\mathfrak{g}$ is a pair $(\mathfrak{U}(\mathfrak{g}), i)$, which satisfy the following:

(1) $\mathfrak{U}(\mathfrak{g})$ is an associative algebra with unit over $\mathbb{F}$.

(2) $i : \mathfrak{g} \to \mathfrak{U}(\mathfrak{g})$ is linear and $i([x, y]) = i(x)i(y) - i(y)i(x)$, for all $x, y \in \mathfrak{g}$.

(3) (Universal Property) For any associative algebra A with unit over $\mathbb{F}$ and for any linear map $j : \mathfrak{g} \to A$ satisfying $j([x, y]) = j(x)j(y) - j(y)j(x)$ for each $x, y \in \mathfrak{g}$, there exists a unique homomorphism of algebras $\theta : \mathfrak{U}(\mathfrak{g}) \to A$ such that $\theta \circ i = j$

Moreover, we can say that the following diagram commutes.

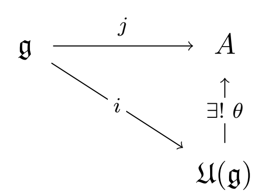


Figure 3.1: Universal Property of UEA

Since $\mathfrak{g}$ is any Lie algebra there is no guarantee that $\mathfrak{g}$ has associative multiplication. Note that the Lie bracket is not necessarily the commutator, however, applying i to the bracket of any two $x, y \in \mathfrak{g}$ must give the commutator of $i(x)$ and $i(y)$. As an aside we should note that Definition 3.1.1 does not require $\mathfrak{g}$ to be of finite dimension or over a field with a particular characteristic. This leaves us with a possible construction of $\mathfrak{U}(\mathfrak{g})$ for which $\mathfrak{g}$ is infinite dimensional.

Theorem 3.1.1 (Uniqueness and Existence of $\mathfrak{U}(\mathfrak{g})$)**.** *If $\mathfrak{g}$ is any Lie algebra over an arbitrary field $\mathbb{F}$, then $(\mathfrak{U}(\mathfrak{g}), i)$ exists and is unique, up to isomorphism.*

Proof. (Uniqueness) We prove this in the normal convention in that we suppose that the Lie algebra $\mathfrak{g}$ has two universal enveloping algebras $(\mathfrak{U}(\mathfrak{g})), i)$ and $(\mathfrak{B}(\mathfrak{g}), i')$. By definition, for each associative $\mathbb{F}$-algebra A there exists a unique homomorphism $\varphi_A : \mathfrak{U}(\mathfrak{g}) \to A$. In particular, since $\mathfrak{B}(\mathfrak{g})$ is an associative $\mathbb{F}$-algebra, we have a unique homomorphism of algebras $\phi : \mathfrak{U}(\mathfrak{g}) \to \mathfrak{B}(\mathfrak{g})$. Moreover, we can, by similar logical progression, reverse the roles of $\mathfrak{U}$ and $\mathfrak{B}$; then there must exist a unique homomorphism of algebra $\psi : \mathfrak{B}(\mathfrak{g}) \to \mathfrak{U}(\mathfrak{g})$. Then $\phi \circ \psi$ factors through $\mathfrak{U}(\mathfrak{g})$ as well as the unit of $\mathfrak{U}(\mathfrak{g})$, namely $1_{\mathfrak{U}(\mathfrak{g})}$; therefore, by uniqueness $\phi \circ \psi = 1$. Now consider $\psi \circ \phi$. This mapping factors through $\mathfrak{B}(\mathfrak{g})$ and $1_{\mathfrak{B}(\mathfrak{g})}$. Again, by uniqueness of the homomorphism we know $\psi \circ \phi = 1$, which implies that ϕ is invertible and thus a bijection. However, ϕ was already a unique homomorphism, therefore it is an isomorphism. Thereby making $(\mathfrak{U}(\mathfrak{g}), i)$ unique, up to isomorphism.

(Existence)[Hum97, Ch. 17] Let $\mathcal{T}(\mathfrak{g})$ be the tensor algebra on $\mathfrak{g}$ and J be the two sided ideal in $\mathcal{T}(\mathfrak{g})$ generated by all the elements of the form $x \otimes y - y \otimes x - [x, y]$, where x and y are in $\mathfrak{g}$. Now we define $\mathfrak{U}(\mathfrak{g})$ to be $\mathcal{T}(\mathfrak{g})/J$ (i.e $\mathfrak{U}(\mathfrak{g}) = \mathcal{T}(\mathfrak{g})/J$ and consider the canonical homomorphism $\pi : \mathcal{T}(\mathfrak{g}) \to \mathfrak{U}(\mathfrak{g})$. So, we can observe that

$$J \subset \bigoplus_{k=1}^{\infty} T^k \mathfrak{g}.$$

Consequently, π maps $T^0\mathfrak{g} = \mathbb{F}$ isomorphically into $\mathfrak{U}(\mathfrak{g})$, which guarantees $\mathfrak{U}(\mathfrak{g})$ contains at minimum scalars. Now we claim that $(\mathfrak{U}(\mathfrak{g}), i)$ is in fact a universal enveloping algebra of $\mathfrak{g}$, where $i : \mathfrak{g} \to \mathfrak{U}(\mathfrak{g})$ is the restriction of π to $\mathfrak{g} \subset \mathcal{T}(\mathfrak{g})$. Let A be any unital associative algebra over the field $\mathbb{F}$ and $j : \mathfrak{g} \to A$ be a linear map satisfying the condition $j([x, y]) = j(x)j(y) - j(y)j(x)$ for all $x,\ y \in \mathfrak{g}$. The universal property of the tensor algebra supplies us with a unique algebra homomorphism $\varphi : \mathcal{T}(\mathfrak{g}) \to A$ that extends j and maps 1 to 1. Because j is defined in a way that imposes the Lie algebra's commutator structure on the associative algebra, this forces all the elements of the form $x \otimes y - y \otimes x - [x, y]$ be in the kernel of φ for all $x,\ y \in \mathfrak{g}$. Therefore, φ induces a homomorphism $\theta : \mathfrak{U}(\mathfrak{g}) \to A$ such that $\varphi \circ i = j$. We can see the uniquness of θ, since 1 and the image of i together generate $\mathfrak{U}(\mathfrak{g})$. □

Remark 2. Theorem 3.1.1 reveals to us that $\mathfrak{U}(\mathfrak{g})$ of $\mathfrak{g}$ can be viewed as the *maximal* associative algebra over an arbitrary field with unity generated by $\mathfrak{g}$ satisfying the relation $xy - yx = [x, y]$ for $x, y \in \mathfrak{g}$ [HK02].

The next Lemma and Proposition will give us a look at how the elements in $\mathfrak{U}(\mathfrak{g})$ behave under the left adjoint action in $\mathfrak{g}$. Although there is a strictly inductive proof of this result, we use an important fact about $\mathrm{ad}(x)$ and its behavior in an associative algebra.

Lemma 3.1.2. *Let R_x (and L_x) be right (and left) multiplication by x in an associative algebra, then the actions R_x and L_x commute with each other in an associative algebra.*

Proof. Let A be an associative algebra and $a, b, c \in A$, then

$$L_a R_b(c) = L_a(cb) = a(cb) = (ac)b = R_b(ac) = R_b L_a(c).$$

□

Proposition 3.1.3. *Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{U}(\mathfrak{g})$ be its universal enveloping Lie algebra, then for any $x,\ y \in \mathfrak{U}(\mathfrak{g})$ and $k \in \mathbb{Z}_{\geq 0}$, we have*

$$(\text{ad } x)^k(y) = \sum_{i=0}^{k}(-1)^i\binom{k}{i}x^{k-i}yx^i. \tag{3.1}$$

Proof. By the previous Lemma, we know that since $\mathfrak{U}(\mathfrak{g})$ is an associative algebra, then L_x and R_x commute. Moreover, $\text{ad}(x) = L_x - R_x$ so we can apply the binomial theorem to obtain the following equalities.

$$\begin{aligned}(\text{ad}(x))^k(y) = (L_x - R_x)^k(y) &= \sum_{i=0}^{k}(-1)^k\binom{k}{i}L_x^{k-i}R_x^k(y) \\ &= \sum_{i=0}^{k}(-1)^k\binom{k}{i}L_x^{k-i}(yx^k) \\ &= \sum_{i=0}^{k}(-1)^k\binom{k}{i}x^{k-i}yx^k\end{aligned}$$

Thus concluding our proof. □

3.2 Poincaré-Birkhoff-Witt Theorem

Remark 3. Depending upon the textbook from which you are studying, there are different variations of what the author may refer to as the Poincaré-Birkhoff-Witt Theorem (or PBW Theorem). For the purpose of this paper, we will use the formulation of the theorem found in Chapter 1 of [HK02]. It is interesting, however, to compare how different authors state the PBW Theorem. For example, in [Hum97] the PBW Theorem is defined as an isomorphism between a *symmetric* algebra and a *graded* associative algebra. Moreover, the way the the same theorem is stated in this paper is really a collection of two corollaries of what [Hum97] states the PBW-Theorem to be.

Theorem 3.2.1 (Poincaré-Birkhoff-Witt Theorem)**.** *[HK02]*

(i) The map $i : \mathfrak{g} \to \mathfrak{U}(\mathfrak{g})$ is injective.

(ii) Let $\{x_\alpha | \alpha \in \Omega\}$ be an ordered basis of $\mathfrak{g}$. Then, all the elements of the form $x_{\alpha_1}x_{\alpha_2}\cdots x_{\alpha_n}$ satisfying $\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_n$ together with 1 form a basis of $\mathfrak{U}(\mathfrak{g})$.

A proof of the PBW Theorem can be found, with great detail and fluidity, in Chapter 17 of [Hum97] and Chapter five of [Jac79].

Part (1) of the Theorem 3.2.1 shines some light on how we can identify each $g \in \mathfrak{g}$ with $i(g) \in \mathfrak{U}(g)$, thereby allowing us to think of $\mathfrak{U}(\mathfrak{g})$ as a larger algebra "enveloping" $\mathfrak{g}$. The next example shows how we can construct a basis of $\mathfrak{U}(\mathfrak{g})$ using the PBW Theorem. Bases of these type are often called PBW-type bases.

Definition 3.2.1 (Polynomial Algebra)**.** Let $\mathbb{F}$ be a field. The **polynomial algebra** on n indeterminates $X_1, X_2, \ldots, X_n$ is the algebra that is spanned by all the linear combinations over $\mathbb{F}$ of products of the commuting variables $X_i, 1 \leq i \leq n$. This algebra is denoted $\mathbb{F}[X_i]$.

Definition 3.2.2 (Symmetric Algebra)**.** The **symmetric algebra** $S(V)$ on a vector space V over a field $\mathbb{F}$ is the free commutative unital associative algebra over $\mathbb{F}$ containing V.

Lemma 3.2.2. *Let $\mathbb{F}[X_i]$ be a polynomial algebra and $S(V)$ be a symmetric algebra. When* $\dim(V)$ *is equal to the number of indeterminates of $\mathbb{F}[X_i]$, then $\mathbb{F}[X_i] \cong S(V)$.*

Example 3.2.1. Let $\mathfrak{g}$ be an abelian Lie algebra of dimension 2 with basis $\{x_1, x_2\}$ over the field $\mathbb{F}$. We know that the bracket $[x_1, x_2] = 0$. So defining the relations of the elements in the basis to be $X_1X_2 - X_2X_1 = 0$, then by Theorem 3.2.1, we know that all the elements of the form $X_1^a X_2^b$ where $a, b \in \mathbb{Z}_{\geq 0}$ together with 1 form a basis of $\mathfrak{U}(\mathfrak{g})$. But since the relationship yields symmetry of the elements under multiplication, we have that $\mathfrak{U}(\mathfrak{g})$ is symmetric and therefore isomorphic to the polynomial algebra of two variables by Lemma 3.2.2.

We can extend this to the n-dimensional case for an abelian Lie algebra.

Example 3.2.2. Let $\mathfrak{g}$ be an abelian Lie algebra of dimension n with a basis $\{x_1, x_2 \ldots, x_n\}$ over the field $\mathbb{F}$. Again, because $\mathfrak{g}$ is abelian, we have that $\forall_{i,\ j}\ [x_i, x_j] = 0$. This tells us that the basis elements of $\mathfrak{U}(\mathfrak{g})$ have the relationship that $\forall_{1 \leq i \leq j \leq n}\ X_iX_j - X_jX_i = 0$. So, the elements in $\mathfrak{U}(\mathfrak{g})$ form a symmetric algebra that is isomorphic to the polynomial algebra of n variables. This is inductively extended from the two dimensional case. So, in light of this result we can view all n-dimensional Lie algebras as a polynomial algebra in n variables.

It is interesting to note that the choice of ordering on the basis elements of $\mathfrak{g}$ is arbitrary. Up to a different labeling, the PBW-type basis is the same. In construction of these types of bases, the ordering of the basis is imposed, rather than a specific ordering being required.

3.3 Representations of $\mathfrak{U}(\mathfrak{g})$

Definition 3.3.1. A **representation** of an associative algebra on a vector space V is an algebra homomorphism $\varphi : A \to \operatorname{End} V$.

Like in the case of a Lie algebra, a representation of an associative algebra over a field with unity on a vector space defines a module structure on the vector space and vice versa.

Theorem 3.3.1. *A representation of $\mathfrak{g}$ can be extended naturally to a representation of $\mathfrak{U}(\mathfrak{g})$. If we let φ be a Lie algebra homomorphism and $\bar{\varphi}$ be an associative algebra homomorphism, then the following diagram commutes. Note: By "restrict" we mean that we are only considering the elements from End(V) for which $[a, b] = ab - ba$ holds.*

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\ \varphi\ } & gl(V) \\ \Big\downarrow{\scriptstyle \text{Univ. Prop}} & & \Big\uparrow{\scriptstyle \text{Restrict}} \\ \mathfrak{U}(\mathfrak{g}) & \xrightarrow{\ \bar{\varphi}\ } & \operatorname{End}(V) \end{array}$$

Proof. As with representations of Lie algebras, a representation of an associative algebra over a vector space defines a module structure on the vector space, and vice versa. To this end, consider a $\mathfrak{g}$-module, say $\mathcal{V}$, on V, and let $g_1 g_2 g \cdots g_n$ be an element from $\mathfrak{U}(\mathfrak{g})$. We can define the action of $\mathfrak{U}(\mathfrak{g})$ on V by

$$(g_1 g_2 g_3 \cdots g_n) \cdot v = g_1 \cdot ((g_2 g_3 \cdots g_n) \cdot v) = \cdots = g_1 \cdot (g_2 \cdot (g_3 \cdots (g_n \cdot v)))$$

for all $g_1, g_2, \ldots, g_n \in \mathfrak{g}, v \in V$. Since $\mathfrak{U}(\mathfrak{g})$ is generated by $\mathfrak{g}$ (see Remark 2) and $g_1 \cdot (g_2 \cdot (g_3 \cdots (g_n \cdot v)))$ determines an action of $\mathfrak{U}(\mathfrak{g})$ on V.

Now, suppose $\mathcal{V}$ is a $\mathfrak{U}(\mathfrak{g})$-module. Since elements of $\mathfrak{g}$ can be identified as elements of $U(\mathfrak{g})$ using the injective mapping we get from part (1) of the PBW theorem, then $\mathcal{V}$ is also a $\mathfrak{g}$-module. Moreover, we have shown that $\mathcal{V}$ can be treated as a $\mathfrak{U}(\mathfrak{g})$-module and $\mathfrak{g}$-module simultaneously; thus, there is a natural extension from representations of $\mathfrak{g}$ to representations of $\mathfrak{U}(\mathfrak{g})$ and vice versa. □

An alternate way of wording Theorem 3.3.1 is given in [EW06] and written below.

Theorem 3.3.2. *Let $\mathfrak{g}$ be a Lie algebra and let $\mathfrak{U}(\mathfrak{g})$ be its universal enveloping algebra. There is a bijective correspondence between $\mathfrak{g}$-modules and $\mathfrak{U}(\mathfrak{g})$-modules.*

The proof given in [EW06] uses the authors construction of $\mathfrak{U}(\mathfrak{g})$, which differs at length from the one given in Definition 3.1.1 of this paper. As where we have defined $\mathfrak{U}(\mathfrak{g})$ by its universal property, Erdmann and Wildon have not. Therefore, this result to them proves that $\mathfrak{U}(\mathfrak{g})$ has a universal property as where our construction of $\mathfrak{U}(\mathfrak{g})$ imposes this property on the associative algebra. It should also be noted that this bijective correspondence between modules gives us a faithful representation from $\mathfrak{g}$ to $\mathfrak{U}(\mathfrak{g})$. So when we consider the universal enveloping algebra as a representation of $\mathfrak{g}$, there is no collapse of any important information pertaining to $\mathfrak{g}$.

Chapter 4

Kac-Moody Lie Algebras

In the 1960s, Victor Kac and Robert Moody began working on Lie algebras that were not of finite dimension. Moody "construct[ed] the Lie algebras, derive[d] their basic properties, and construct[ed] a symmetric invariant from on those Lie algebras derived from so-called symmetrizable generalized Cartan matrices" in his paper *A New Class of Lie Algebras*, published in the Journal of Algebra [BP02]. Although working on a similar question, on the other side of the world, Victor Kac was devoted to extending the construction which Jacobson had presented in Chapter 7 to the infinite-dimensional setting [BP02]. Combining both Moody's and Kac's finding (along with their names) we have Kac-Moody Lie algebras. The simplest examples of infinite-dimensional Kac-Moody Lie algebras are those of affine type and are the major focus of this paper.

4.1 Basic Definitions and Constructions

The definitions defined in this chapter are based on [Kac94, HK02], chapters one and two respectively. We will include a few proofs, but refer the read to [Kac94] for further elaboration on particular results. To begin this section we will look at an important matrix which encodes the information of any Lie algebra. For all definitions henceforth in this chapter, let I be a finite index set.

Definition 4.1.1. A square matrix $A = a_{ij}$ $(i, j \in I)$ with the following defining relations is called a **Cartan matrix**:

(C1) $a_{ii} = 2$,

(C2) $a_{ij} \in \mathbb{Z}_{\leq 0}$ if $i \neq j$,

(C3) $a_{ij} = a_{ji} \Leftrightarrow a_{ji} = 0$,

(C4) Each proper principal minor of A is positive.

Definition 4.1.2. If there exists a diagonal matrix $D = \mathrm{diag}(s_i | i \in I)$ with all $s_i \in \mathbb{Z}_{\geq 0}$ such that DA is symmetric, then A is said to be **symmetrizable**.

Remark 4. For the purpose of this paper, we will assume that the generalized Cartan matrix A is symmetrizable.

Definition 4.1.3. The (generalized) Cartan matrix A is said to be **indecomposable** if for every pair of nonempty subsets $I_1,\ I_2 \subset I$ with $I_1 \cup I_2 = I$, there exists some $i \in I_1$ and $j \in I_2$ such that $a_{ij} \neq 0$.

Definition 4.1.4. The **dual weight lattice**, denoted $P^\vee$, is a free abelian group of rank $2|I| - \text{rank } A$ with an integral basis

$$\{h_i | i \in I\} \cup \{d_s | s = 1, \dots, |I| - \text{rank } A\}.$$

We can also define the **weight space** to be

$$P = \{\lambda \in \mathfrak{h}^* | \lambda(P^\vee) \in \mathbb{Z}\}.$$

Definition 4.1.5. The **Cartan subalgebra**, denoted $\mathfrak{h}$, is the $\mathbb{F}$-linear space spanned by $P^\vee$ such that $\mathfrak{h} = \mathbb{F} \otimes_{\mathbb{Z}} P^\vee$.

Definition 4.1.6. Let $\Pi^\vee = \{h_i | i \in I\}$ and choose a linearly independent subset $\Pi = \{\alpha_i | i \in I\} \subset \mathfrak{h}^*$ satisfying

$$\alpha_j(h_i) = a_{ij}, \quad \alpha_j(d_s) = 0 \text{ or } 1$$

for $i, h \in I, s = 1, \dots, |I| - \text{rank } A$. The elements of Π and $\Pi^\vee$ are called **simple roots** and **simple coroots** respectively.

Definition 4.1.7. Let $\Lambda_i \in \mathfrak{h}^*$, where $i \in I$, be the linear functionals on $\mathfrak{h}$ given by

$$\Lambda_i(h_i) = \delta_{ij}, \quad \Lambda_i(d_s) = 0, \text{ for } j \in I, s = 1, \dots, |I| - \text{rank } A.$$

The elements Λ_i are called the **fundamental weights**.

The triple $(\mathfrak{h}, \Pi, \Pi^\vee)$ is a **realization** of A and the quintuple $(A, \Pi, \Pi^\vee, P, P^\vee)$ is called the **Cartan datum** associated with the generalized Cartan matrix A. The free abelian group $Q = \bigoplus_{i \in I} \mathbb{Z}\alpha_i$ is called the **root lattice** and $Q_+ = \sum_{i \in I} \mathbb{Z}_{\geq 0}\alpha_i$ is called the **positive root lattice**. Equivalently we say the negative root lattice is $Q_- = -Q_+$. There is a partial ordering on $\mathfrak{h}^*$ defined by $\lambda \geq \mu$ if and only if $\lambda - \mu \in Q_+$ for $\lambda, \mu \in \mathfrak{h}^*$.

Definition 4.1.8. A **Kac-Moody algebra** $\mathfrak{g}$ associated with a Cartan datum $(A, \Pi, \Pi^\vee, P, P^\vee)$ is the Lie algebra generated by the elements $e_i,\ f_i$ for each $i \in I$ and $h \in P^\vee$ subject to the following defining relations:

(KM1) $[h, h'] = 0$ for $h,\ h \in P^\vee$,

(KM2) $[e_i, f_i] = \delta_{ij} h_i$,

(KM3) $[h, e_i] = \alpha_i(h) e_i$ for $h \in P^\vee$,

(KM4) $[h, f_i] = -\alpha_i(g) f_i$ for $h \in P^\vee$,

(KM5) $(\text{ad } e_i)^{1 - a_{ij}} e_j = 0$ for $i \neq j$,

(KM6) $(\text{ad } f_i)^{1 - a_{ij}} f_j = 0$ for $i \neq j$.

Remark 5. Relations (1)-(4) are called the Weyl relations and (5)-(6) are called the Serre relations. The subalgebra generated by e_i we denote to be $\mathfrak{n}_+$ and the subalgebra generated by f_i we denote to be $\mathfrak{n}_-$. Moreover, for each $\alpha \in Q$, let

$$\mathfrak{g}_\alpha = \{x \in \mathfrak{g} |\ [h, x] = \alpha(h)(x) \text{ for all } h \in \mathfrak{h}\}.$$

This next theorem outlines properties of Kac-Moody algebra and are proved in [Kac94].

Theorem 4.1.1. *Let $\mathfrak{g}$ be a Kac-Moody algebra with an associated generalized Cartan matrix A, then the following statements are true.*

(i) $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$, *which is called the* ***triangular decomposition****.*

(ii) The subalgebras $\mathfrak{n}_+$ and $\mathfrak{n}_-$ coupled with the defining relations (KM5) and (KM6) respectively in Definition 4.1.8 are generated by e_i and f_i respectively.

(iii) With respect to $\mathfrak{h}$ we have the ***root space decomposition****:*

$$\mathfrak{g}(A) = \left(\bigoplus_{0 \neq \alpha \in Q_+} \mathfrak{g}_{-\alpha} \right) \oplus \mathfrak{h} \oplus \left(\bigoplus_{0 \neq \alpha \in Q_+} \mathfrak{g}_{\alpha} \right).$$

Furthermore, $\dim \mathfrak{g} < \infty$*, and* $\mathfrak{g}_\alpha \subset \mathfrak{n}_\pm$ *for* $\pm\alpha \in Q_+, \alpha \neq 0$.

In light of this, $\mathfrak{g}_\alpha = \{x \in \mathfrak{g} | \ [h, x] = \alpha(h)(x) \text{ for all } h \in \mathfrak{h}\}$ is called the **root space** attached to α and $\mathfrak{g}_0 = \mathfrak{h}$. The integer mult $\alpha := \dim \mathfrak{g}_\alpha$ is called the *multiplicity* of α. If α is an element in Q, $\alpha \neq 0$, and mult $\neq 0$ then α is called a *root*. It follows directly from the previous theorem that every root is either positive or negative. Let (1) Δ_+ be the sets of all positive roots then (2) $\Delta_- = -\Delta_+$ are all negative roots and (3) $\Delta = \Delta_+ \dot{\cup} \Delta_-$ is the collection of all roots.

Invoking the power of Proposition 3.1.3, we obtain the following theorem summarizing the generators and relations for $\mathfrak{U}(\mathfrak{g})$.

Theorem 4.1.2. *The universal enveloping algebra $\mathfrak{U}(\mathfrak{g})$ of $\mathfrak{g}$ is the associative algebra over $\mathbb{F}$ with unity generated by $e_i, \ f_i \ (i \in I)$ and $\mathfrak{h}$ subject to the following defining relations:*

(i) $hh' = h'h$ *for* $h, \ h' \in \mathfrak{h}$,

(ii) $e_i f_j - f_j e_i = \delta_{ij} h_i$ *for* $i, \ j \in I$,

(iii) $he_i - e_i h = \alpha(h) e_i$ *for* $h \in \mathfrak{h}, \ i \in I$,

(iv) $hf_i - f_i h = -\alpha(h) f_i$ *for* $h \in \mathfrak{h}, \ i \in I$,

(v) $\displaystyle\sum_{k=0}^{1-a_{ij}} (-1)^k \binom{1 - a_{ij}}{k} e_i^{1-a_{ij}-k} e_j e_i^k = 0$ *for* $i \neq j$,

(vi) $\displaystyle\sum_{k=0}^{1-a_{ij}} (-1)^k \binom{1 - a_{ij}}{k} f_i^{1-a_{ij}-k} f_j f_i^k = 0$ *for* $i \neq j$,

Proof. The proof of this theorem is straightforward: the construction of $\mathfrak{U}$ supplies us with verification of *(i)-(iv)* and Proposition 3.1.3 verifies the rest. □

4.2 Generalized Cartan Matrices and Dynkin Diagrams

In this section we will talk briefly about the classifications of the generalized Cartan matrix. There are three classifications of a generalized Cartan matrix: *finite*, *affine*, and *indefinite*. Each of these classifications have particular defining features and are discussed and proven in chapter four of [Kac94]. For the purpose of our discussion, we will only consider generalized Cartan matrices of the affine type. If A is an indecomposable generalized Cartan matrix and of affine type, then the following are true about A:

(1) the corank, which is the number of rows minus the rank, of A is 1;

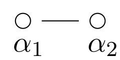
(2) there exists a $u > 0$ such that $Au = 0$;

(3) if $Av \geq 0$, then $Av = 0$.

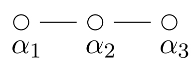
For each A, we can associate to it a graph called the **Dynkin diagram**. The construction of such diagram is as follows. If $a_{ij}a_{ji} \leq 4$ and $|a_{ij}| \geq |a_{ji}|$, the vertices i and j are connect by $|a_{ij}|$ lines, and these lines contain an arrow pointing toward i if $|a_{ij}| > 1$. If $a_{ij}a_{ji} > 4$, the vertices i and j are connected by a bold-faced line with an ordered pair of integers $(|a_{ij}|, |a_{ji}|)$.

Example 4.2.1. Consider the following Cartan matrices and their corresponding Dynkin diagrams:

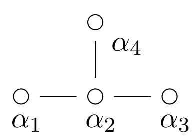
(1) $\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$



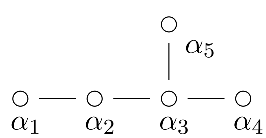
(2) $\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$



(3) $\begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & -1 \\ 0 & -1 & 2 & 0 \\ 0 & -1 & 0 & 2 \end{pmatrix}$



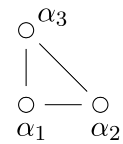
(4) $\begin{pmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & -1 \\ 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 0 & 2 \end{pmatrix}$



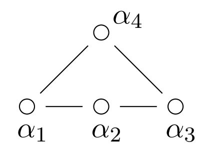
Examples (1) and (2) are of finite type and correspond to root systems which we label A_2 and A_3 respectively. Examples (3) and (4) are also of finite type and correspond to root systems which we label D_4 and D_5 respectively. We can immediately begin to see that D_4 is the smallest of the finite type D because D_3 would correspond to A_3. Furthermore, this would give us isomorphisms of the corresponding Lie algebras.

Example 4.2.2. Now we shall consider Dynkin diagrams corresponding to affine generalized Cartan matrices.

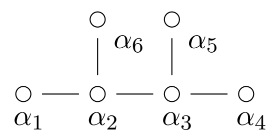
(1) $$\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$$



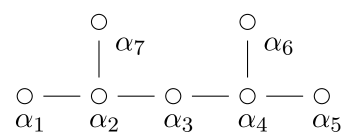
(2) $$\begin{pmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{pmatrix}$$



(3) $$\begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & -1 \\ 0 & -1 & 2 & 0 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 0 & 2 & 0 \\ 0 & -1 & 0 & 0 & 0 & 2 \end{pmatrix}$$



(4) $$\begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & -1 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & -1 & 2 & -1 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 2 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}$$



In this example (1) and (2) correspond to root systems labeled $A_2^{(1)}$ and $A_3^{(1)}$ respectively. Labels for (3) and (4) are $D_5^{(1)}$ and $D_6^{(1)}$ respectively. These two types of affine Kac-Moody algebras will be the concern of this paper, and denoted, in general, as $A_n^{(1)}$ and $D_n^{(1)}$. We read this as "a n upper one" and equivalently for D.

Remark 6. Kac-Moody algebras can be thought of as being "one off" from a finite Lie algebra in both their corresponding generalized Cartan matrix and Dynkin diagram. This can be observed from their generalized Cartan matrices and Dynkin diagrams. Consider (1) from Example 4.2.2, then we can see in the following matrix, that the shaded blue area is the Cartan matrix for A_2, which is "embedded" inside $A_2^{(1)}$.

$$\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$$

Furthermore, we can see that a Dynkin diagrams corresponding to a finite Lie algebra is a sub-diagram of a Dynkin diagram corresponding to a affine Lie algebra. For a look at all the Dynkin diagrams for both finite and affine Lie algebras, see Appendix B.

4.3 Representation theory of Kac-Moody algebras

The theory presented here should mirror the theory for $\mathfrak{sl}_2$ presented in Chapter 2. However, we must adapt the definitions and constructions by minor changes in notation.

Definition 4.3.1. A $\mathfrak{g}$-module is called a **weight module** if it admits the following weight space decomposition:

$$V = \bigoplus_{\lambda \in \mathfrak{h}^*} V_\lambda, \text{ where } V_\lambda = \{v \in V \mid hv = \lambda(h)v \text{ for all } h \in \mathfrak{h}\}. \tag{4.1}$$

(1) A module is said to be *diagonalizable* if it has the weight space decomposition from Equation 4.1.

(2) A vector $v \in V_\lambda$ is called a **weight vector** of weight λ.

(3) A vector v is called a **maximal vector** of weight λ if $e_i v = 0$ for all $i \in I$.

(4) If $V_\lambda \neq 0$, then λ is a **weight** of V and V_λ is the **weight space** associated with λ.

(5) The dimension of V_λ is called the **weight multiplicity** of λ. The set of weights for a $\mathfrak{g}$-module is denoted $\mathrm{wt}(V)$.

Remark 7. In general, if we are given an abelian group M, then a decomposition $V = \bigoplus_{\beta \in M} V_\beta$ of the vector space V into a direct sum of its subspaces is called an *M-gradation* of V. A subspace $U \subset V$ is called *graded* if $U = \bigoplus_{\beta \in M}(U \cap V_\beta)$. The elements from V_β are called *homogeneous* of degree β.

Theorem 4.3.1. *Let $\mathfrak{h}$ be a commutative Lie algebra and V a diagonalizable $\mathfrak{h}$-module. Then any submodule of V is graded with respect to the gradation 4.1.*

Proof. Let U be a submodule of V. Any $v \in V$ can be written in the form $v = \sum_{i=1}^n v_i$, where $v_i \in V_{\lambda_i}$. Here each v_i occupies a distinct eigenspace and are therefore linearly independent. Let $\mathrm{span}\{v_i \mid i = 1, \dots, n\}$ be such that the collection of v_i's are a basis for U, that we will denote β. Moreover, there exists $h \in \mathfrak{h}$ such that $\lambda_i(h)$ is distinct for all i in the index set $\{1, \dots, n\}$. So, for $v \in U$, we can write

$$h^k(v) = \sum_{i=1}^n \lambda_i(h)^k v_i, \qquad \text{for all } k = 0, 1, \dots, n-1.$$

Since these are invariant vector spaces, then $h^k(v)$ is an element in U for each k. Then the coordinate matrix for $[v]_\beta$ and $h^k(v)$ would be respectively

$$[v]_\beta = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}, \qquad [h^k(v)]_\beta = \begin{bmatrix} \lambda_1^k \\ \vdots \\ \lambda_n^k \end{bmatrix}.$$

The collection of $h^k(v)$ create a coefficient matrix for a system of linear equations constructed in the following fashion:

$$[[v]_\beta \mid [h(v)]_\beta \mid \ldots \mid [h^{n-1}(v)]_\beta] = \begin{bmatrix} 1 & \lambda_1 & \cdots & \lambda_1^{n-1} \\ 1 & \lambda_2 & \cdots & \lambda_2^{n-1} \\ \vdots & \vdots & & \vdots \\ 1 & \lambda_n & \cdots & \lambda_n^{n-1} \end{bmatrix}.$$

The resulting matrix is a Vandermonde matrix since each λ_i is distinct, and therefore this matrix is nonsingular. If v_i is a unit vector $(i \times 1)$ with a one in the ith position, then for the system of equation we can say

$$\begin{bmatrix} 1 & \lambda_1 & \cdots & \lambda_1^{n-1} \\ 1 & \lambda_2 & \cdots & \lambda_2^{n-1} \\ \vdots & \vdots & & \vdots \\ 1 & \lambda_n & \cdots & \lambda_n^{n-1} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{n-1} \end{bmatrix} = v_i.$$

From this we know that the system is solvable in U for all v_i, where $i \in \{1, \ldots, n\}$. From this, we can conclude that all v_i lie in U. □

Definition 4.3.2. The **category** $\mathcal{O}$ is the collection of weight modules V over $\mathfrak{g}$ with finite dimensional weight spaces for which there exists a finite number of elements $\lambda_1, \lambda_2, \ldots, \lambda_s \in \mathfrak{h}^*$ such that

$$\mathrm{wt}(V) \subset D(\lambda_1) \cup \cdots \cup D(\lambda_s),$$

where $D(\lambda) = \{\mu \in \mathfrak{h}^* \mid \mu \leq \lambda \text{ for } \lambda \in \mathfrak{h}^*\}$.

The morphisms are $\mathfrak{g}$-module homomorphisms. Moreover, it can be shown that the category $\mathcal{O}$ is closed under the finite direct sum (or finite tensor product) of objects from the category $\mathcal{O}$; the quotients of $\mathfrak{g}$-modules from the category $\mathcal{O}$ are also in the category $\mathcal{O}$.

Definition 4.3.3. A weight module V is a **highest weight module** of **highest weight** $\lambda \in \mathfrak{h}^*$ if there exists a nonzero vector $v_\lambda \in V$, called a **highest weight vector**, such that

(1) $e_i v_\lambda = 0$ for all $i \in I$,

(2) $h v_\lambda = \lambda(h) v_\lambda$ for all $h \in \mathfrak{h}$

(3) $V = \mathfrak{U}(\mathfrak{g}) v_\lambda$ (which necessarily implies that $V = \mathfrak{U}(\mathfrak{g})_- v_\lambda$, the Lie subalgebra generated by the f_i's in $\mathfrak{U}(\mathfrak{g})$).

The highest weight module is an interesting example for $\mathfrak{g}$-modules in the category $\mathcal{O}$ that we shall consider.

Definition 4.3.4. Let $\mathfrak{g}$ be a Lie algebra and V and $\mathfrak{g}$-module. We say that x in $\mathfrak{g}$ is **locally nilpotent** on V if for any $v \in V$ there exists a positive integer N such that $x^N \cdot v = 0$.

Lemma 4.3.2. *Let $\mathfrak{g}$ be a Lie algebra and V be a $\mathfrak{g}$-module.*

(i) Let $\{y_i \mid i \in \Lambda\}$ be a set of generator of $\mathfrak{g}$ and let $x \in \mathfrak{g}$. If for every $i \in \Lambda$ there exists a positive integer N_i such that $(\mathrm{ad}\ )^{N_i}(y_i) = 0$, then $\mathrm{ad}\ x$ is locally nilpotent on $\mathfrak{g}$.

(ii) Let $\{v_i |\ i \in \Lambda\}$ be a set of generators of V and let $x \in \mathfrak{g}$. If for each $i \in \Lambda$ there exists a positive integer N_i such that $x^{N_i} \cdot v_i = 0$ and $\operatorname{ad} x$ is locally nilpotent on $\mathfrak{g}$, then x is locally nilpotent on V.

Proof. [Kac94, Ch. 4] To prove this result we rely on the *Leibnitz formula* for derivations, namely for $x, y, z \in \mathfrak{g}$ and $N > 0$:

$$(\operatorname{ad} x)^N([x, z]) = \sum_{i=0}^{N} \binom{N}{k} [(\operatorname{ad} x)^k(y), (\operatorname{ad} x)^{N-k}(z)], \tag{4.2}$$

$$x^N y = \sum_{i=0}^{N} \binom{N}{k} ((\operatorname{ad} x)^k(y) x^{N-k}). \tag{4.3}$$

Equation (4.3) is associated with the universal enveloping algebra where $(\operatorname{ad} x)(y) = xy - yx$. By induction over the y_i's we yield the result we wished to prove. □

Definition 4.3.5. A weight module V over a Kac-Moody algebra is called **integrable** if all e_i and f_i $(i \in I)$ are locally nilpotent on V.

Definition 4.3.6. The **category** $\mathcal{O}_{\text{int}}$ consists of integrable $\mathfrak{g}$-modules in the category $\mathcal{O}$ such that $\text{wt}(V) \subset P$.

Remark 8. A result of this definition is that any $\mathfrak{g}$-module V in the category $\mathcal{O}_{\text{int}}$ has a weight space decomposition

$$V = \bigoplus_{\lambda \in \mathfrak{h}^*} V_\lambda, \text{ where } V_\lambda = \{v \in V \mid hv = \lambda(h)v \text{ for all } h \in P^\vee\}.$$

If we fix $i \in I$, then $\mathfrak{g}_{(i)}$ is the subalgebra of $\mathfrak{g}$ generated by $e_i,\ f_i,\ h_i$. From this we obtain that $\mathfrak{g}_{(i)}$ is isomorphic to $\mathfrak{sl}_2$. The same can be said for $\mathfrak{U}(\mathfrak{g})$, denoted $\mathfrak{U}_{(i)}$.

As a result of Lemma 4.3.2, a highest weight $\mathfrak{g}$-module with highest weight λ and highest weight vector v_λ is integrable if and only if for every $i \in I$, there exists a $N_i \in \mathbb{Z}_{\geq 0}$ such that $f_i^{N_i} v_\lambda = 0$.

Definition 4.3.7. Consider the following set of weights:

$$P^+ = \{\lambda \in P |\ \lambda(h_i) \in \mathbb{Z}_{\geq 0} \text{ for all } i \in I\}.$$

The weights of this set are called **dominant integral weights**.

Lemma 4.3.3. *[Kac94, Ch.10]*

(i) Let $V(\lambda)$ be the irreducible highest weight g-module with highest weight $\lambda \in \mathfrak{h}^*$. Then $V(\lambda)$ is in the category $\mathcal{O}_{int}$ if and only if $\lambda \in P^+$.

(ii) Every irreducible $\mathfrak{g}$-module in the category $\mathcal{O}_{int}$ is isomorphic to $V(\lambda)$ for some $\lambda \in P^+$

Theorem 4.3.4. *[Kac94, Ch.10] Let $\mathfrak{g}$ be a Kac-Moody algebra associated with a Cartan datum $(A, \Pi, \Pi^\vee, P, P^\vee)$. Then every $\mathfrak{g}$-module in the category $\mathcal{O}_{int}$ is isomorphic to a direct sum of irreducible highest weight modules $V(\lambda)$ with $\lambda \in P^+$.*

Corollary 4.3.5. *[Kac94, Ch.10] The tensor product of a finite number of $\mathfrak{g}$-modules in the category $\mathcal{O}_{int}$ is completely reducible.*

We leave the proofs of these to be referenced by the reader, but mention them to bring closure to this chapter on Kac-Moody algebras. The next chapter will develop the necessary information on *quantum groups*. However, before we move on, we should note that the results presented in this chapter are an extremely small subset of those from the field of Kac-Moody algebras. A more deep and rich theory on Kac-Moody algebras and their representations can be found in [Kac94]. These results are the bare minimum to study the crystal basis theory we wish to achieve in this text.

Chapter 5

Quantum Groups

In this chapter, we will develop a basic understanding of Quantum Groups. The notion of a *quantum group* rises from the idea that quantum mechanics is a *deformation* of classical mechanics. The nomenclature in Quantum Group Theory parallels that of the quantum mechanics, but from a mathematical perspective. In spite of their name, quantum groups are not groups at all but rather an associative (albeit noncommutative) algebra with unit. The construction of this algebra arises from imposing a deformation parameter on the universal enveloping algebra of a Kac-Moody algebra. The theory that we developed in the previous chapter on Kac-Moody algebras and their representation will carry over to the quantum group setting with a change in notation. Moreover, the material presented here is a summary of that presented in [HK02, Ch. 3]

5.1 Quantum groups

Fix an indeterminate q. We call this the quantum parameter. We will begin by defining how elements in the scalar field in the quantum group are realized.

Definition 5.1.1. An elements $[n]_q$ is called a q-**integer**, and is written in the form

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}. \tag{5.1}$$

As in the usual sense of probability, we define $[0]_q! = 0$ and $[n]_q! = [n]_q \cdot [n-1]_q \cdots [1]_q$ for $n \in \mathbb{Z}_{>0}$. Moreover, let $n \geq m \neq 0$, then the quantum equivalence to the binomial coefficients are given by

$$\begin{bmatrix} n \\ m \end{bmatrix}_q = \frac{[n]_q!}{[n]_q![n-m]_q!}, \tag{5.2}$$

and are called q-**binomial coefficients**.

Moreover, $[n]_q$ and $\begin{bmatrix} n \\ m \end{bmatrix}_q$ are elements of the field $\mathbb{F}(q)$, which is a field of quotients with the fixed indeterminate.

Again, let A be a symmetrizable generalized Cartan matrix with a symmetrizing matrix $D = \operatorname{diag}(s_i \in \mathbb{Z}_{>0} |\ i \in I)$ and let $(A, \Pi, \Pi^\vee, P, P^\vee)$ be a Cartan datum associated with A.

Definition 5.1.2. The **quantum group** or the **quantized universal enveloping algebra** $\mathfrak{U}_q(\mathfrak{g})$ associated with a Cartan datum $(A, \Pi, \Pi^\vee, P, P^\vee)$ is the associative algebra over

$\mathbb{F}(q)$ with 1 generated by the elements $e_i,\ f_i\ (i \in I)$ and $q^h\ (h \in P^\vee)$ with the following defining relations:

(1) $q^0 = 1,\ q^h q^{h'} = q^{h+h'}$ for $h, h' \in P^\vee$,

(2) $q^h e_i q^{-h} = q^{\alpha_i(h)} e_i$ for $h \in P^\vee$,

(3) $q^h f_i q^{-h} = q^{-\alpha_i(h)} f_i$ for $h \in P^\vee$,

(4) $e_i f_i - f_i e_i = \delta_{ij} \frac{q^{s_i h_i} - q^{-s_i h_i}}{q^{s_i} - q^{-s_i}}$ for $i, j \in I$,

(5) $\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1 - a_{ij} \\ k \end{bmatrix}_{q^{s_i}} e_i^{1-a_{ij}-k} e_j e_i^k = 0$ for $i \neq j$,

(6) $\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1 - a_{ij} \\ k \end{bmatrix}_{q^{s_i}} f_i^{1-a_{ij}-k} f_j f_i^k = 0$ for $i \neq j$.

Relations (5) and (6) should be familiar to the reader as they are the quantum analog to the Serre relations and are thus called the *quantum Serre relations*. Set $\deg f_i = -\alpha_i$, $\deg q^h = 0$, and $\deg e_i = \alpha_i$. The α root space is given by

$$(\mathfrak{U}_q)_\alpha = \{u \in \mathfrak{U}_q(\mathfrak{g}) |\ q^h u q^{-h} = q^{\alpha(h)} u \text{ for all } h \in P^\vee\}. \tag{5.3}$$

Since all the defining relations of the quantum group $\mathfrak{U}_q(\mathfrak{g})$ are homogeneous, it exhibits a **root space decomposition**

$$\mathfrak{U}_q(\mathfrak{g}) = \bigoplus_{\alpha \in Q} (\mathfrak{U}_q)_\alpha. \tag{5.4}$$

Similarly, it can be shown that the *quantum adjoint operator* has the following identity:

$$(\mathrm{ad}_q\ e_i)^N (e_j) = \sum_{k=0}^{N} (-1)^k q_i^{k(N+a_{ij}-1)} \begin{bmatrix} N \\ k \end{bmatrix}_{q_i} e_i^{N-k} e_j e_i^k.$$

This identity allows us to write the quantum Serre relations in a more familiar form:

$$(\mathrm{ad}_q\ e_i)^{1-a_{ij}} (e_j) = 0, \qquad (\mathrm{ad}_q\ f_i)^{1-a_{ij}} (f_j) = 0 \text{ for } i \neq j.$$

Example 5.1.1. Consider the quantum group $U_q(\mathfrak{sl}_2)$, which is generated by the elements $e,\ f$, and $q^{\pm h}$ as defined in Example 4.2.1 in [HK02, Ch 4]. The defining relations of this quantum group are

$$q^h e q^{-h} = q^2 e, \quad q^h f q^{-h} = q^{-2} f, \quad ef - fe = \frac{q^h - q^{-h}}{q - q^{-1}}. \tag{5.5}$$

Moreover, this quantum group has a two-dimensional representation of $V = \mathbb{F}(q)v_{-1} \oplus \mathbb{F}(q)v_1$ and module action defined by

$$v_1 \begin{cases} e \cdot v_1 = 0 \\ f \cdot v_1 = v_{-1} \\ q^h \cdot v_1 = q v_1 \end{cases} \qquad v_{-1} \begin{cases} e \cdot v_{-1} = v_1 \\ f \cdot v_{-1} = 0 \\ q^h \cdot v_{-1} = q^{-h} v_{-1} \end{cases}.$$

We refer to this representation as the vector representation.

5.2 Representation theory of quantum groups

The theory presented here should look strikingly familiar to the representation theory of Kac-Moody algebras.

Definition 5.2.1. A $\mathfrak{U}_q(\mathfrak{g})$-module V^q is called a **weight module** if it admits the following weight space decomposition:

$$V^q = \bigoplus_{\lambda \in P} V^q_\lambda, \text{ where } V^q_\lambda = \{v \in V^q | \; q^h v = q^{\lambda(h)} v \text{ for all } h \in P^\vee\}. \tag{5.6}$$

(1) A vector $v \in V^q_\lambda$ is called a **weight vector** of weight λ.

(2) A vector v is called a **maximal vector** of weight λ if $e_i v = 0$ for all $i \in I$.

(3) If $V^q_\lambda \neq 0$, then λ is a **weight** of V^q and V^q_λ is the **weight space** associated with $\lambda \in P$.

(4) The dimension of V^q_λ is called the **weight multiplicity** of λ. The set of weights for a $\mathfrak{U}_q(\mathfrak{g})$-module is denoted $\mathrm{wt}(V)$.

Definition 5.2.2. The **category** $\mathcal{O}^q$ is the collection of weight modules V^q over $\mathfrak{U}_q(\mathfrak{g})$ with finite dimensional weight spaces for which there exists a finite number of elements $\lambda_1, \lambda_2, \ldots, \lambda_s \in \mathfrak{h}^*$ such that

$$\mathrm{wt}(V^q) \subset D(\lambda_1) \cup \cdots \cup D(\lambda_s),$$

where $D(\lambda) = \{\mu \in P | \; \mu \leq \lambda \text{ for } \lambda \in P\}$.

Just as in the case with Kac-Moody algebras, the most important examples among the $\mathfrak{U}_q(\mathfrak{g})$-modules is the category $\mathcal{O}^q$ for our discussion is the *highest weight modules*. A weight module V^q is called a **highest weight module** with **highest weight** weight $\lambda \in P$ if there exists a nonzero $v_\lambda \in V^q$ such that

$$\begin{aligned} e_i v_\lambda &= 0 \text{ for all } i \in I, \\ q^h v_\lambda &= q^{\lambda(h)} v_\lambda \text{ for all } h \in P^\vee, \\ V^q &= \mathfrak{U}_q(\mathfrak{g}) v_\lambda, . \end{aligned}$$

The third equation implies that $V^q = \mathfrak{U}_q(\mathfrak{g})_- v_\lambda$, which can be generated by the f_i elements in $\mathfrak{U}_q(\mathfrak{g})$. The vector v_λ is called the **highest weight vector** and is unique up to a constant multiple.

Definition 5.2.3. The **category** $\mathcal{O}^q_{\mathrm{int}}$ consists of $\mathfrak{U}_q(\mathfrak{g})$-modules V^q satisfying the following conditions:

(1) V^q has a weight space decomposition $V^q = \bigoplus_{\lambda \in P} V^q_\lambda$, where

$$V^q_\lambda = \{v \in V^q | \; q^h v = q^{\lambda(h)} v \text{ for all } h \in P^\vee\}$$

and the dimension of V^q_λ is finite for all $\lambda \in P$,

(2) there exists a finite number of elements $\lambda_1, \dots, \lambda_s \in P$ such that

$$\mathrm{wt}(V^q) \subset D(\lambda_1) \cup \cdots \cup D(\lambda_s),$$

(3) all e_i and f_i ($i \in I$) are locally nilpotent on V^q.

The morphisms in this collection are assumed to be the usual $\mathfrak{U}_q(\mathfrak{g})$-module homomorphisms.

Remark 9. The category $\mathcal{O}^q_{\text{int}}$ consists of integrable $\mathfrak{U}_q(\mathfrak{g})$-modules in the category $\mathcal{O}^q$. Moreover, $\mathcal{O}^q_{\text{int}}$ is closed under direct sums and tensor products of finitely many $\mathfrak{U}_q(\mathfrak{g})$-modules.

Now we will shift the focus and consider the localization of $\mathbb{F}[q]$ at the ideal $(q-1)$:

$$\mathbf{A}_1 = \{f(q) \in \mathbb{F}[q] |\ f \text{ is regular at } q = 1\} \tag{5.7}$$
$$= \{g/h|\ g,\ h \in \mathbb{F}[q],\ h(1) \neq 0\}. \tag{5.8}$$

Definition 5.2.4. Let $n \in \mathbb{Z}$, then we define the following relations:

$$[y;n]_q = \frac{yq^n - y^{-1}q^{-n}}{q - q^{-1}} \quad \text{and} \quad (y;n)_q = \frac{yq^n - 1}{q-1}.$$

The next lemma shows where these relations live with respect to the quantum group elements.

Lemma 5.2.1. *Let $\mathfrak{U}^+_{\boldsymbol{A}_1}$ be the $\boldsymbol{A}_1$-subalgebra of $\mathfrak{U}_{\boldsymbol{A}_1}$ generated by the elements e_i for $i \in I$; let $\mathfrak{U}^-_{\boldsymbol{A}_1}$ be the $\boldsymbol{A}_1$-subalgebra of $\mathfrak{U}_{\boldsymbol{A}_1}$ generated by the elements f_i for $i \in I$; and let $\mathfrak{U}^0_{\boldsymbol{A}_1}$ be the $\boldsymbol{A}_1$-subalgebra of $\mathfrak{U}_{\boldsymbol{A}_1}$ generated by the elements q^h for $h \in P^\vee$. The the following are true:*

(i) $(q^h; n)_q \in \mathfrak{U}^0_{\boldsymbol{A}_1}$ *for all* $n \in \mathbb{Z}$ *and* $h \in P^\vee$.

(ii) $[q^{s_i h_i}; n]_q \in \mathfrak{U}^0_{\boldsymbol{A}_1}$ *for all* $n \in \mathbb{Z}$ *and* $i \in I$.

Definition 5.2.5. The $\mathbf{A}_1$**-form**, denoted $\mathfrak{U}_{\mathbf{A}_1}$ generated by the elements e_i and f_i respectively for $i \in I$. Moreover, let $\mathfrak{U}^0_{\mathbf{A}_1}$ be the $\mathbf{A}_1$-subalgebra of $\mathfrak{U}_{\mathbf{A}_1}$ generated by q^h

5.3 Classical Limit

In this section V^q will denote a highest weight $\mathfrak{U}_q(\mathfrak{g})$-module of highest weight $\lambda \in P$ and highest weight vector v_λ. Let $\mathbf{J}_1$ be the unique maximal ideal of the local ring $\mathbf{A}_1$ generated by $q-1$. Then there exists an isomorphism of fields defined by

$$\mathbf{A}_1/\mathbf{J}_1 \xrightarrow{\sim} \mathbb{F} \text{ given by } f(q) + \mathbf{J}_1 \mapsto f(1).$$

Remark 10. Under this mapping, the quantum parameter q is mapped onto 1.

We define the $\mathbb{F}$-linear vector spaces

$$\mathfrak{U}_1 = (\mathbf{A}_1/\mathbf{J}_1) \otimes_{\mathbf{A}_1} \mathfrak{U}_{\mathbf{A}_1}, \tag{5.9}$$
$$V^1 = (\mathbf{A}_1/\mathbf{J}_1) \otimes_{\mathbf{A}_1} V_{\mathbf{A}_1}. \tag{5.10}$$

As a result of this definition, V^1 is a $\mathfrak{U}_1$-module. It can also be shown that $\mathfrak{U}_1$ is isomorphic to $\mathfrak{U}(\mathfrak{g})$ and that V^1 is a highest weight $\mathfrak{U}(\mathfrak{g})$-module of highest weight λ. To do this, we begin by realizing that

$$\mathfrak{U}_1 \cong \mathfrak{U}_{\mathbf{A}_1}/\mathbf{J}_1\mathfrak{U}_{\mathbf{A}_1} \text{ and } V^1 \cong V_{\mathbf{A}_1}/\mathbf{J}_1 V_{\mathbf{A}_1}.$$

Then the natural maps

$$\begin{aligned} \mathfrak{U}_{\mathbf{A}_1} &\to \mathfrak{U}_{\mathbf{A}_1}/\mathbf{J}_1\mathfrak{U}_{\mathbf{A}_1} \cong \mathfrak{U}_1, \\ V_{\mathbf{A}_1} &\to V_{\mathbf{A}_1}/\mathbf{J}_1 V_{\mathbf{A}_1} \cong V^1. \end{aligned}$$

We will use the usual notation of placing a bar over the image of these maps. The taking of these maps are called "taking the *classical limit*". Again, q is mapped to 1 under these mappings. The following theorem gives a nice result of the how the elements under this mapping behave.

Lemma 5.3.1. *For each $\mu \in P$, define $V^1_\mu = (\boldsymbol{A}_1/\boldsymbol{J}_1) \otimes_{\boldsymbol{A}_1} (V_{\boldsymbol{A}_1})_\mu$. The the following are true:*

(i) For each $\mu \in P$, if $\{v_i\}$ is a basis of the free $\boldsymbol{A}_1$-module $(V_{\boldsymbol{A}_1})_\mu$, then $\{\bar{v}_i\}$ is a basis of the $\mathbb{F}$-linear space V^1_μ.

(ii) For each $\mu \in P$, a set $\{v_i\} \subset (V_{\boldsymbol{A}_1})_\mu$ is $\boldsymbol{A}_1$-linearly independent if the set $\{\bar{v}_i\} \subset V^1_\mu$ is $\mathbb{F}$-linearly independent.

Theorem 5.3.2.

(i) The elements $\bar{e}_i$, $\bar{f}_i$ $(i \in I)$ and $\bar{h}$ $(h \in P^\vee)$ satisfy the defining relations of $\mathfrak{U}(\mathfrak{g})$ given by Proposition 4.1.8. Therefore, there exists a surjective $\mathbb{F}$-algebra homomorphism $\varphi : \mathfrak{U}(\mathfrak{g}) \to U_1$ and the $\mathfrak{U}_1$-module V^1 has a $\mathfrak{U}(\mathfrak{g})$-module structure.

(ii) For each $\mu \in P$ and $h \in P^\vee$, the elements $\bar{h}$ acts on V^1 as scalar multiplication by $\mu(h)$. Thus V^1_μ is the μ-weight space of the $\mathfrak{U}(\mathfrak{g})$-module V^1.

(iii) As a $\mathfrak{U}(\mathfrak{g})$-module, V^1 is a highest weight module with highest weight $\lambda \in P$ and highest weight vector $\bar{v}_\lambda$.

Remark 11. We would like to conclude this section by clarifying the construction of the classical limit by way of the unique maximal ideal of a local ring by a quotient. If we let $\mathbf{J}_n$, where $n \in \mathbb{Z}^{\geq 0}$, be the unique maximal ideal of the local ring $\mathbf{A}_n$ generated by $q - n$, then the quotient $\mathbf{A}_n/\mathbf{J}_n$ will map the elements generated by $q-n$ to zero in $\mathbb{F}$. This means that $q = n$. For the classical limit, this is precisely $q = 1$. Moreover, this is exactly what we would expect if we were to take the limit as q goes to one in the traditional calculus sense of limits. What we have done in this chapter is define the limit in an algebraic fashion. This definition of the limit will help us when we discuss the *crystal limit* in the next chapter.

Chapter 6

Crystal Bases Theory

Crystal bases were introduced by Masaki Kashiwara (1947 -) and George (Gheorghe) Lusztig (1946 -) in 1990. A *crystal base*, or *canonical base* as it was first presented, is a base of a representation, such that generators of a quantum group or semisimple Lie algebra have particularly simplistic yet useful actions called the *Kashiwara operators*. These bases are realizations of the quantum groups when $q = 0$ and inherit numerous combinatorial features that reflect the internal structure of integrable representations of quantum groups in the category $\mathcal{O}^q_{\text{int}}$. The definitions and theorems from this chapter are taken from [HK02] and should be referenced for further elaboration on this subject.

6.1 Motivating Example

Before we delve into the theory, lets begin my looking at Example 4.2.1 again from [HK02, Ch. 4] that demonstrates why we would like to develop crystal bases for quantum groups.

Example 6.1.1. Let V be a highest weight $\mathfrak{U}_q(\mathfrak{sl}_2)$-module with highest weight vector v_1 and weight of 1. We would now like to consider the tensor product $V \otimes V$. An apparent basis of this module would be

$$v_1 \otimes v_1, \quad v_1 \otimes v_{-1}, \quad v_{-1} \otimes v_1, \quad v_{-1} \otimes v_{-1}.$$

However, when $q \neq 0$, it is does not correspond to the irreducible decomposition of $V \otimes V$, namely $V \otimes V \cong V(2) \oplus V(0)$. So, let's begin by looking at how the tensors behave under the action of f. The module action on a tensor product (extended linearly) is defined as

$$x \cdot (a \otimes b) = (x \cdot a) \otimes b + a \otimes (x \cdot b). \tag{6.1}$$

The weights of corresponding weight vectors are added together under the tensor product. Therefore, the highest weight vector in the tensor product above is $v_1 \otimes v_1$. Let's look at the action of f on this vector. Recalling the module actions from Example 5.1.1, we get the following diagram for the action of f.

$$
\begin{array}{c}
v_1 \otimes v_1 \\
\Big\downarrow f \\
v_{-1} \otimes v_1 + v_1 \otimes v_{-1} \\
\Big\downarrow f \\
v_{-1} \otimes v_{-1} + v_{-1} \otimes v_{-1}
\end{array}
$$

Figure 6.1

From this, we can see that the weight of the tensors are 2, 0, and -2 respectively from the top-down. This would be the basis for $V(2)$, thereby leaving $v_1 \otimes v_{-1}$ to be a basis for $V(0)$. However, this is not the only possible basis for V, but is one of them. Another basis, given by [HK02, Ch.4], is

$$v_1 \otimes v_1, \quad v_1 \otimes v_{-1} - qv_{-1} \otimes v_1, \quad v_{-1} \otimes v_1 + qv_1 \otimes v_{-1}, \quad v_{-1} \otimes v_{-1}.$$

For this particular basis, the natural basis and the quantum basis are equivalent when $q = 0$, which is a desirable relationship. Moreover, this would lead us to suspect that there are particular bases for $\mathfrak{U}_q(\mathfrak{g})$-modules in the category $\mathcal{O}^q_{\text{int}}$ that have similar desirable behavior at $q = 0$.

6.2 Kashiwara operators

We will develop the crystal basis theory from $\mathfrak{U}_q(\mathfrak{g})$-modules in the category $\mathcal{O}^q_{\text{int}}$. We will begin by stating a useful lemma in the development and defining relations of the *Kashiwara operators.*

Lemma 6.2.1. *Let M^q be a $\mathfrak{U}_q(\mathfrak{g})$-module in the category $\mathcal{O}^q_{int}$ where $M = \bigoplus_{\lambda \in P} M_\lambda$. Then for each $i \in I$, every weight vector $u \in M_\lambda$ $(\lambda \in \mathrm{wt}(M))$ may be written in the form*

$$u = u_0 + f_i u_1 + f_i^{(2)} u_2 + \cdots + f_i^{(N)} u_N,$$

where $N \in \mathbb{Z}_{\geq 0}$ and $u_k \in M_{\lambda + k\alpha_i} \cap \ker e_i$ for all $k = 0, 1, \ldots, N$.

Each u_k in the expression is uniquely determined by u and $u_k \neq 0$ only if $\lambda(h_i) + k \geq 0$.

Remark 12. Here we see that every weight vector can be written as a sum of weight vectors living inside different weight spaces. Moreover, each u_k must be an element in a weight space $M_{\lambda + k\alpha_i}$ and is mapped to zero under the action of e_i, i.e. $u_k \in \ker e_i$.

Definition 6.2.1. The **Kashiwara operators**, denoted $\tilde{e}_i$ and $\tilde{f}_i$ $(i \in I)$ on M^q, are defined as follows

$$\tilde{e}_i u = \sum_{k=1}^{N} f_i^{(k-1)} u_k, \qquad \tilde{f}_i u = \sum_{k=0}^{N} f_i^{(k+1)} u_k. \tag{6.2}$$

Let's look at an example of how we can use this lemma and consequently, the Kashiwara operators on a basis element from Example 6.1.1.

Example 6.2.1. We will divide this example in to two part. The first will be decomposing a weight vector into the sum of weight vectors. The second part will consist of using the definition of Kashiwara operators on the given weight vector.

(1) Consider the vector $v_{-1} \otimes v_1$. We wish to write this vector in terms of a finite sum of weight vectors as in Lemma 6.2.1. To this end, we must find u_0: a zero weight vector that becomes zero when acted upon by e. Using the element $v_1 \otimes v_{-1} - qv_{-1} \otimes v_1$, we see that this is in the zero weight space and the action of e sends it to zero; therefore some multiple of $v_1 \otimes v_{-1} - qv_{-1} \otimes v_1$ is our u_0. To find u_1 we need a vector in M_2^q that is sent to zero under the action of e. Therefore, u_1 is equal to some multiple of $v_1 \otimes v_1$ under the action of f, which we know is $v_{-1} \otimes v_1 + qv_1 \otimes v_{-1}$. We should note that when we say multiple, we mean that we take the coefficient to be from the polynomial ring $\mathbb{F}[q]$.

$$\begin{aligned} v_{-1} \otimes v_1 &= f(q)(\underbrace{v_1 \otimes v_{-1} - qv_{-1} \otimes v_1}_{u_0}) + g(q)(f \cdot (\underbrace{v_1 \otimes v_1}_{u_1})) \\ &= f(q)(v_1 \otimes v_{-1} - qv_{-1} \otimes v_1) + g(q)(v_{-1} \otimes v_1 + qv_1 \otimes v_{-1}) \end{aligned}$$

To finish this part, we need to solve the system of equation for $f(q)$ and $g(q)$

$$\begin{aligned} -qf(q) + g(q) &= 1, \\ f(q) + qg(q) &= 0. \end{aligned}$$

With some algebraic manipulation, we arrive at

$$f(q) = -\frac{q}{q^2+1}, \quad g(q) = \frac{1}{q^2+1},$$

and

$$v_{-1} \otimes v_1 = -\frac{q}{q^2+1}(v_1 \otimes v_{-1} - qv_{-1} \otimes v_1) + \frac{1}{q^2+1}(v_{-1} \otimes v_1 + qv_1 \otimes v_{-1}).$$

(2) Now, consider again $v_{-1} \otimes v_1 = f(q)(v_1 \otimes v_{-1} - qv_{-1} \otimes v_1) + g(q)(f \cdot (v_1 \otimes v_1))$ defined in this way. Then

$$\begin{aligned} \tilde{e}_i(v_{-1} \otimes v_1) &= g(q)(f^0 u_1) \\ &= g(q)(v_1 \otimes v_1) \end{aligned}$$

$$\begin{aligned} \tilde{f}_i(v_{-1} \otimes v_1) &= g(q)(f \cdot u_0 + f^{(2)} \cdot u_1) \\ &= g(q)(f \cdot (v_1 \otimes v_{-1} - qv_{-1} \otimes v_1) + f^{(2)}(v_1 \otimes v_1)) \\ &= g(q)(\frac{1}{2} f \cdot (f \cdot (v_1 \otimes v_1))) \\ &= g(q)(v_{-1} \otimes v_{-1}) \end{aligned}$$

We see that when $q = 0$ we have $g(q) = 1$, the natural basis discussed in Example 6.1.1 is our exact result and the Kashiwara operators move the vectors in the most natural way possible.

Proposition 6.2.2. *The following describes the basic properties of Kashiwara operators. Again, let M^q be a $\mathfrak{U}_q(\mathfrak{g})$-module in the category $\mathcal{O}^q_{int}$ where $M = \bigoplus_{\lambda \in P} M_\lambda$. Then*

(i) $\tilde{e}_i M_\lambda = e_i M_\lambda \subset M_{\lambda+\alpha_i}$, $\quad \tilde{f}_i M_\lambda = f_i M_\lambda \subset M_{\lambda-\alpha_i}$ *for all* $i \in I$ *and* $\lambda \in P$.

(ii) The Kashiwara operators $\tilde{e}_i$ *and* $\tilde{f}_i$ *commute with* $\mathfrak{U}_q(\mathfrak{g})$*-module homomorphisms.*

Reverting back to the localization $\mathbf{A}_1$, we now consider $\mathbf{A}_0$ of the polynomial ring $\mathbb{F}[q]$ at the ideal (q):

$$\begin{aligned} \mathbf{A}_0 &= \{f(q) \in \mathbb{F}(q) |\ f \text{ is regular at } q = 0\} \\ &= \{g/h|\ g, h \in \mathbb{F}[q],\ h(0) \neq 0\}. \end{aligned}$$

This results in the local ring $\mathbf{A}_0$ to be a principle ideal domain with $\mathbb{F}(q)$ as its field of quotients. This setup should be familiar as it was the driving relation to develop the classical limit. The same will be done here, but with crystals.

Definition 6.2.2. Let M^q be a $\mathfrak{U}_q(\mathfrak{g})$-module in the category $\mathcal{O}^q_{\text{int}}$. A free $\mathbf{A}_0$-submodule $\mathcal{L}$ of M^q is called a **crystal lattice** if

(1) $\mathcal{L}$ generates M^q as a vector space over $\mathbb{F}(q)$ i.e $\mathbb{F}(q) \otimes_{\mathbf{A}_0} \mathcal{L}_\lambda \cong M^q_\lambda$ for each $\lambda \in \text{wt}(M)$,

(2) $\mathcal{L} = \bigoplus_{\lambda \in P} \mathcal{L}_\lambda$, where $\mathcal{L} = \mathcal{L} \cap M_\lambda$ for all $\lambda \in P$,

(3) $\tilde{e}_i \mathcal{L} \subset \mathcal{L}$, $\tilde{f}_i \mathcal{L} \subset \mathcal{L}$ for all $i \in I$.

Now let $\mathbf{J}_0$ be the unique maximal ideal of the local ring $\mathbf{A}_0$ generated by q. Then there exists an isomorphism of fields defined by

$$\mathbf{A}_0 \xrightarrow{\sim} \mathbb{F} \text{ given by } f(q) + \mathbf{J}_0 \mapsto f(0).$$

This forces q to be mapped onto 0 and we have

$$\mathbb{F} \otimes_{\mathbf{A}_0} \mathcal{L} \xrightarrow{\sim} \mathcal{L}/\mathbf{J}_0\mathcal{L} = \mathcal{L}/q\mathcal{L}.$$

The taking of this mapping is referred to as the *crystal limit*. Again, we can think of this construction as in sending q to zero as we would with a normal limit (see Remark 11). As we did with the classical limit, we denote the elements in the image of the crystal limit as $\bar{v}$. Moreover, the Kashiwara operators preserve the lattice $\mathcal{L}$ and define equivalent operators on the quotient $\mathcal{L}/q\mathcal{L}$.

Definition 6.2.3. A **crystal basis** of $\mathfrak{U}_q(\mathfrak{g})$-module M^q in the category $\mathcal{O}^q_{\text{int}}$ is a pair $(\mathcal{L}, \mathcal{B})$ where the following are true:

(1) $\mathcal{L}$ is a crystal lattice of M^q,

(2) $\mathcal{B}$ is an $\mathbb{F}$-basis of $\mathcal{L}/q\mathcal{L} \cong \mathbb{F} \otimes_{\mathbf{A}_0} \mathcal{L}$,

(3) $\mathcal{B} = \dot{\cup}_{\lambda \in P} \mathcal{B}_\lambda$, where $\mathcal{B}_\lambda = \mathcal{B} \cap (\mathcal{L}_\lambda / q\mathcal{L}_\lambda)$,

(4) $\tilde{e}_i \mathcal{B} \subset \mathcal{B} \cup \{0\}$, $\tilde{f}_i \mathcal{B} \subset \mathcal{B} \cup \{0\}$ for all $i \in I$,

(5) for any $b, b' \in \mathcal{B}$ and $i \in I$, we have $\tilde{e}_i b = b'$ if and only if $b = \tilde{e}_i b'$.

Now take $\mathcal{B}$ to be the set of vertices and define the I-colored arrows on $\mathcal{B}$ by

$$b \longrightarrow b' \text{ if and only if } \tilde{f}_i b = b' \ (i \in I).$$

This give $\mathcal{B}$ an I-colored oriented graph structure called the **crystal graph** of M^q. We should also know that this definition of a crystal graph shows that the graph defines the relation of the Kashiwara operators and vice versa.

The following example shows the use of the crystal lattice and crystal graph of a familiar quantum group, $\mathfrak{U}_q(\mathfrak{sl}_2)$.

Example 6.2.2. As we say from Example 5.1.1, $\mathfrak{U}_q(\mathfrak{sl}_2)$ is generated by the elements $e,\ f,\ q^{\pm h}$ with the defining relations given in Equation 5.5. For $m \in \mathbb{Z}_{\geq 0}$, let $V(m)$ be the $(m+1)$-dimensional irreducible $\mathfrak{U}_q(\mathfrak{sl}_2)$-module with a basis $\{u, fu, \ldots, f^{(m)}\}$, where

$$eu = 0, \quad Ku = q^m u,$$

$$f^{(k)}u = \frac{1}{[k]_q!} f^k u \ (k = 0, 1, \ldots, m).$$

Define $\mathcal{L}(m)$ and $\mathcal{B}(m)$ as

$$\mathcal{L}(m) = \bigoplus_{k=0}^{m} \mathbf{A}_0 f^{(k)} u, \quad \mathcal{B}(m) = \{\bar{u}, , \overline{fu}, \ldots, \overline{f^{(m)}u}\}.$$

In this definition of $\mathcal{L}(m)$ and $\mathcal{B}(m)$, u denotes the highest weight vector of weight m. Then we can verify that $(\mathcal{L}(m), \mathcal{B}(m))$ is in fact a crystal basis. This is straightforward and is a run-through of the definitions.

Let $\mathcal{B}$ be the set of vertices and define the I-colored arrows on $\mathcal{B}$ by

$$b \longrightarrow b' \text{ if and only if } \tilde{f}_i b = b' \quad (i \in I).$$

By this relation, $\mathcal{B}$ is given an I-colored oriented graph structure, which we call the **crystal graph** of M^q. We may also refer to this is an i-string. So, if we consider the previous example, namely $\mathcal{B}(m)$, then the crystal graph of $\mathcal{B}(m)$ is

$$\mathcal{B}(m): \quad \overline{u} \longrightarrow \overline{fu} \longrightarrow \cdots \longrightarrow \overline{f^{(m)}u}.$$

This next example will be important for our study of $A_n^{(1)}$.

Example 6.2.3. Consider $U_q(\mathfrak{sl}_n)$. This is a quantum group generated by the element $e_i,\ f_i,\ q^{\pm s_i h_i}$ $(i = 1, \ldots, n-1)$ with the quantum group defining relations from Definition 5.1.2 begin satisfied. Let $V = \bigoplus_{j=1}^{n} \mathbb{F}(q) v_j$ be the n-dimensional vector space over

$\mathbb{F}(q)$ and a basis $\{v_1, \dots, v_n\}$. We define the action of the $\mathfrak{U}_q(\mathfrak{sl}_n)$-module on V by

$$e_i v_j = \begin{cases} v_i & \text{if } j = i+1, \\ 0 & \text{otherwise,} \end{cases}$$

$$f_i v_j = \begin{cases} v_{i+1} & \text{if } j = i, \\ 0 & \text{otherwise,} \end{cases}$$

$$q^{\pm h} v_j = \begin{cases} q^{\pm 1} v_i & \text{if } j = i, \\ q^{\mp 1} v_{i+1} & \text{if } j = i+1, \\ 0 & \text{otherwise.} \end{cases}$$

Under this action V becomes an irreducible highest weight module over $U_q(\mathfrak{sl}_n)$ with highest weight ϵ_1 and highest weight vector v_1. Since this is a highest weight module of $U_q(\mathfrak{sl}_n)$, we give it a special name, the **vector representation**. If we define $\mathcal{L} = \bigoplus_{j=1}^{n} \mathbf{A}_0 v_j$ and $\mathcal{B} = \{\overline{v_1}, \dots, \overline{v_n}\}$, where $\overline{v_j}$ denotes the image of v_j under the crystal limit. It is easily verified that $(\mathcal{L}, \mathcal{B})$ is a crystal basis of V with the crystal graph of $\mathcal{B}$ defined as

$$\mathcal{B}: \quad \overline{v_1} \xrightarrow{1} \overline{v_2} \xrightarrow{2} \cdots \xrightarrow{n-1} \overline{v_n}$$

6.3 Tensor Product Rule

Definition 6.3.1. [HK02, Ch.4] Let $M = \bigoplus_{\lambda \in P} M_\lambda$ be a $\mathfrak{U}_q(\mathfrak{g})$-module in the category $\mathcal{O}^q_{\text{int}}$ with a crystal basis $(\mathcal{L}, \mathcal{B})$. For $i \in I$ and $b \in \mathcal{B}_\lambda$ $(\lambda \in P)$, we define the maps, $\varepsilon_i, \varphi_i : \mathcal{B} \to \mathbb{Z}$ by

$$\varepsilon_i(b) = \max\{k \geq 0 |\ \tilde{e}_i^k b \in \mathcal{B}\},$$
$$\varphi_i(b) = \max\{k \geq 0 |\ \tilde{f}_i^k b \in \mathcal{B}\}.$$

Remark 13. From this, with respect to the vector representation, we can think of $\varepsilon_i(b)$ as the number of edges between b and the beginning of its i-string. Moreover, we can think of $\varphi_i(b)$ as the number of edges between b and the end of its i-string.

Theorem 6.3.1. *[Kas95] Let M_j^q be a $\mathfrak{U}_q(\mathfrak{g})$-module in the category $\mathcal{O}^q_{int}$ and let $(\mathcal{L}_j, \mathcal{B}_j)$ be a crystal basis of M_j^q $(j = 1, 2)$. Set $\mathcal{L} = \mathcal{L}_1 \otimes_{\mathbf{A}_0} \mathcal{L}_2$ and $\mathcal{B} = \mathcal{B}_1 \times \mathcal{B}_2$.*

Then the following are true:

(i) $(\mathcal{L}, \mathcal{B})$ is a crystal basis of $M_1^q \otimes_{\mathbb{F}(q)} M_2^q$.

(ii) The action of the Kashiwara operators $\tilde{e}_i$ and $\tilde{f}_i$ $(i \in I)$ on $\mathcal{B}$ is given by

$$\tilde{e}_i(b_1 \otimes b_2) = \begin{cases} \tilde{e}_i b_1 \otimes b_2 & \text{if } \varphi_i(b_1) \geq \varepsilon_i(b_2), \\ b_1 \otimes \tilde{e}_i b_2 & \text{if } \varphi_i(b_1) < \varepsilon_i(b_2), \end{cases}$$

$$\tilde{f}_i(b_1 \otimes b_2) = \begin{cases} \tilde{f}_i b_1 \otimes b_2 & \text{if } \varphi_i(b_1) > \varepsilon_i(b_2), \\ b_1 \otimes \tilde{f}_i b_2 & \text{if } \varphi_i(b_1) \leq \varepsilon_i(b_2). \end{cases}$$

Corollary 6.3.2. *Under the same assumptions of Theorem 6.3.1, the following equalities hold true:*

$$\begin{aligned}
\mathrm{wt}(b_1 \otimes b_2) &= \mathrm{wt}(b_1) + \mathrm{wt}(b_2),\\
\varepsilon_i(b_1 \otimes b_2) &= \max(\varepsilon_i(b_1), \varepsilon_i(b_1) - \langle h_i, \mathrm{wt}(b_1)\rangle),\\
\varphi_i(b_1 \otimes b_2) &= \max(\varphi_i(b_2), \varphi_i(b_1) - \langle h_i, \mathrm{wt}(b_2)\rangle).
\end{aligned}$$

For a proof of Theorem 6.3.1 and Corollary 6.3.2, see [HK02, Ch. 4.4].

Remark 14. To bring these rules to fruition, we outline a few rules for construction the crystal graph.

(1) The top left-most vector is the highest weight vector in the tensor product.

(2) The ordering should begin column first and then the row.

(3) Each node can only have one arrow of each type entering and exiting it any time. This means that no node can have two 2-arrows entering or exiting it. This also restricts us to only having one arrow of each type exiting each node also.

Example 6.3.1. Let V the vector representation of $\mathfrak{U}_q(\mathfrak{sl}_3)$ with crystal graph

$$\mathcal{B}: \quad \overline{v_1} \xrightarrow{1} \overline{v_2} \xrightarrow{2} \overline{v_3}.$$

We wish to then consider V^* with a vector representation defined by

$$\mathcal{B}^*: \quad \overline{v_3^*} \xrightarrow{2} \overline{v_2^*} \xrightarrow{1} \overline{v_1^*}.$$

Using the tensor product rule, we can construct a crystal graph for $\mathcal{B} \otimes \mathcal{B}^*$.

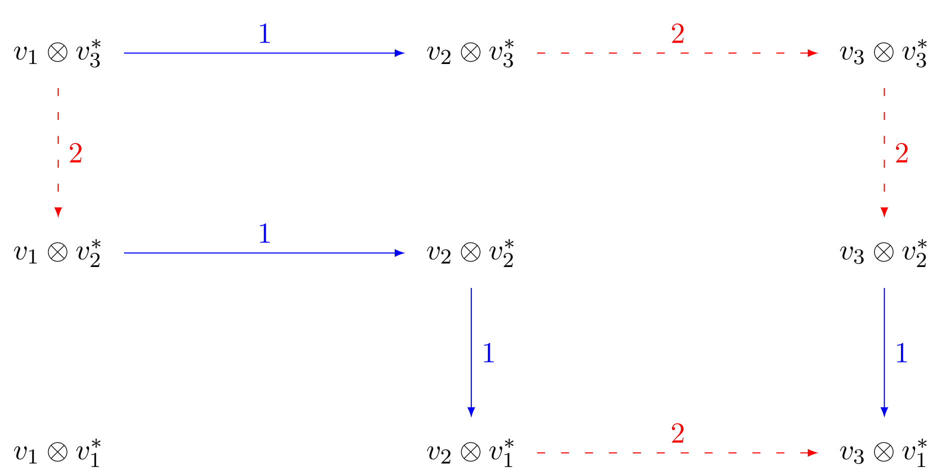


Figure 6.2

6.4 Crystals

We now wish to give the notion of what a *crystal* associated with a Cartan datum is.

Definition 6.4.1. [HK02, Ch.4] Let I be a finite index set and let $A = (a_{ij})_{i,j\in I}$ be a generalized Cartan matrix with Cartan datum $(A, \Pi, \Pi^\vee, P, P^\vee)$. A **crystal** associated with the Cartan datum $(A, \Pi, \Pi^\vee, P, P^\vee)$ is a set $\mathcal{B}$ together with the maps $\mathrm{wt} : \mathcal{B} \to P,\ \tilde{e}_i, \tilde{f}_i : \mathcal{B} \to \mathcal{B} \cup \{0\}$, and $\varepsilon_i, \varphi_i : \mathcal{B} \to \mathbb{Z} \cup \{-\infty\}$ $(i \in I)$ satisfying the following properties:

(1) $\varphi_i(b) = \varepsilon_i(b) + \langle h_i, \mathrm{wt}(b)\rangle$ for all $i \in I$,

(2) $\mathrm{wt}(\tilde{e}_i b) = \mathrm{wt}(b) + \alpha_i$ if $\tilde{e}_i(b) \in \mathcal{B}$,

(3) $\mathrm{wt}(\tilde{f}_i b) = \mathrm{wt}(b) - \alpha_i$ if $\tilde{f}_i(b) \in \mathcal{B}$,

(4) $\varepsilon_i(\tilde{e}_i b) = \varepsilon_i(b) - 1,\ \varphi_i(\tilde{e}_i b) = \varphi_i(b) + 1$ if $\tilde{e}_i(b) \in \mathcal{B}$,

(5) $\varepsilon_i(\tilde{f}_i b) = \varepsilon_i(b) + 1,\ \varphi_i(\tilde{f}_i b) = \varphi_i(b) - 1$ if $\tilde{f}_i(b) \in \mathcal{B}$,

(6) $\tilde{f}_i b = b'$ if and only if $b = \tilde{e}_i b'$ for $b, b' \in \mathcal{B},\ i \in I$,

(7) if $\varphi_i(b) = -\infty$ for $b \in \mathcal{B}$, then $\tilde{e}_i b = \tilde{f}_i b = 0$.

To define the tensor product of two crystals, we simply combine Theorem 6.3.1 and Corollary 6.3.2 into a definition and define the crystal structure based on these relationships.

Remark 15. For a crystal associated with a Cartan datum, we call the basis $\mathcal{B}$ a $\mathfrak{U}_q(\mathfrak{g})$-crystal. Here $\mathfrak{U}_q(\mathfrak{g})$ is the quantum groups associated with the corresponding Cartan datum $(A, \Pi, \Pi^\vee, P, P^\vee)$. Moreover, we let $\mathcal{B}_\lambda = \{b \in \mathcal{B} |\ \mathrm{wt}(b) = \lambda\}$ to ensure that $\mathcal{B} = \dot{\cup}_{\lambda\in P}\mathcal{B}_\lambda$.

6.5 Young Tableaux and Crystals

Now that we have seen the construction of crystal graphs and their rules for construction, we now wish to consider how to generalize this idea. This process would be difficult to visualize as we have done in Example 6.3.1; therefore, we look to another powerful tool to complete this task, namely *young tableau.* First we consider the vector representation of $\mathfrak{gl}_n$, which is strikingly familiar to $\mathfrak{sl}_n$. In fact, the crystal is

$$\mathcal{B}: \quad \boxed{1} \xrightarrow{1} \boxed{2} \xrightarrow{2} \cdots \xrightarrow{n-1} \boxed{n}.$$

As a motivating example, we should consider what the crystal graph of a particular dominant integral weight, say $\lambda = 3\epsilon_1 + 2\epsilon_2 + \epsilon_3$, would look like. To begin, let us define some terms.

Definition 6.5.1. [HK02, Ch. 7.3] A **Young diagram** is a collection of boxes arranged in left-justified rows with a weakly decreasing number of boxes in each row.

Remark 16. There are also Young diagrams that are called *skew* Young diagrams. For these diagrams, the left-justified criteria is omitted.

Definition 6.5.2. [HK02, Ch. 7.3] A **tableau** is a Young diagram filled with numbers, one per box. A **semistandard tableau** is a tableau obtained from a Young diagram where the numbers $1, 2, \ldots, n$ are subject to the following conditions:

(1) the entries in each row are weakly increasing,

(2) the entries in each column are strictly increasing.

For a specific tableau, denoted $\mathcal{T}$, we define its **weight** to be

$$\mathrm{wt}(\mathcal{T}) = k_1\epsilon_1 + \cdots + k_n\epsilon_n,$$

where k_i denotes the number of boxes appearing in the ith row of $\mathcal{T}$.

Example 6.5.1. As an example, we will construct the Young diagram corresponding to $\lambda = 3\epsilon_1 + 2\epsilon_2 + \epsilon_3$ in $\mathfrak{gl}_3$. By Definition 6.5.2, we know that there are three boxes in the first row, two in the second row, and one in the third row. So, the general shape of the tableau is

$$\begin{array}{|c|c|c|}\hline & & \\ \hline & & \multicolumn{1}{c}{} \\ \cline{1-2} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}\,.$$

Definition 6.5.3. [HK02, Ch. 7.3]

(1) The **Far-Eastern reading** of a semistandard tableau is done by reading down a column from top to bottom and then proceeding right to left.

(2) The **Middle-Eastern reading** of a semistandard tableau is done by moving first across the rows from right to left and then proceeding from top to bottom.

Example 6.5.2. Given the tableau below, we wish to expand it into the tensor product of the individual blocks.

$$\begin{array}{|c|c|c|}\hline 1 & 1 & 3 \\ \hline 2 & 3 & \multicolumn{1}{c}{} \\ \cline{1-2} 3 & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}$$

The Far-Eastern Reading would be

$$\begin{array}{|c|c|c|}\hline 1 & 1 & 3 \\ \hline 2 & 3 & \multicolumn{1}{c}{} \\ \cline{1-2} 3 & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} = \boxed{3} \otimes \boxed{1} \otimes \boxed{3} \otimes \boxed{1} \otimes \boxed{2} \otimes \boxed{3}$$

and the Middle-Eastern reading would be

$$\begin{array}{|c|c|c|}\hline 1 & 1 & 3 \\ \hline 2 & 3 & \multicolumn{1}{c}{} \\ \cline{1-2} 3 & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} = \boxed{3} \otimes \boxed{1} \otimes \boxed{1} \otimes \boxed{3} \otimes \boxed{2} \otimes \boxed{3}.$$

So, let us pick the Middle-Eastern reading. If we apply the action of $\tilde{f}_1$, then by using the Tensor Product Rule from Theorem 6.3.1 we get the result

$$\boxed{3} \otimes \boxed{2} \otimes \boxed{1} \otimes \boxed{3} \otimes \boxed{2} \otimes \boxed{3}.$$

Using this method, we can construct the entire crystal graph using only Crystal Basis Theory. However, if we build the corresponding tableau, then we realize that

$$\begin{array}{|c|c|c|}\hline 1 & 1 & 3 \\ \hline 2 & 3 & \multicolumn{1}{c}{} \\ \cline{1-2} 3 & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \xrightarrow{1} \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 2 & 3 & \multicolumn{1}{c}{} \\ \cline{1-2} 3 & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array},$$

which corresponds to the classical Young tableau theory.

Remark 17. It can be shown that regardless of which of the two readings you choose, you can still define a $\mathfrak{U}_q(\mathfrak{gl}_n)$-crystal structure on the set of all semistandard tableaux of a particular shape. Moreover, the readings above are "stable" under Kashiwara operators. For proof and further elaboration on this topic, the reader can refer to [HK02, Ch. 7.3]. This means that we can completely determine the crystal structure of a finite $\mathfrak{U}_q(\mathfrak{gl}_n)$ using young tableaux theory.

Example 6.5.3. Let us retrogress back to the beginning and consider the dominant integral weight $\lambda = 3\epsilon_1 + 2\epsilon_2 + \epsilon_3$ in $\mathfrak{gl}_3$. With these new tools, we can write this as a tensor from the crystal, namely

$$\boxed{1} \otimes \boxed{1} \otimes \boxed{1} \otimes \boxed{2} \otimes \boxed{2} \otimes \boxed{3}$$

and consequently construct the corresponding young tableau to achieve the following

$$\begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline 2 & 2 \\ \cline{1-2} 3 \\ \cline{1-1} \end{array} \, .$$

Since we ultimately wish to draw the $\mathfrak{U}_q(\mathfrak{gl}_3)$-crystal $\mathcal{B}(Y)$ for λ using young tableaux theory, we need some rules on how to complete this task.

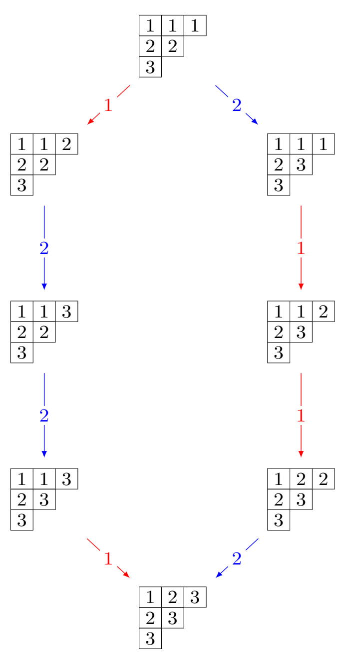
To draw a $\mathfrak{U}_q(\mathfrak{gl})$-crystal using the aforementioned theory, we can use the following rules:

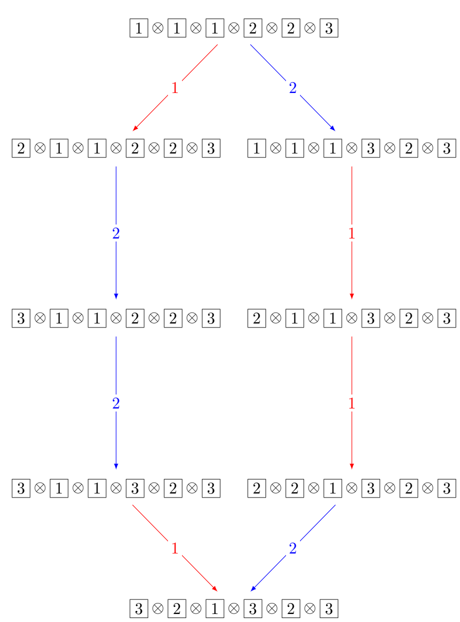
(i) see if any box can be increased by 1 without breaking the criterion from Definition 6.5.2;

(ii) if a box can be increased, draw a new tableau underneath it with the corresponding change made; draw a colored arrow coming from the tableau to the changed tableau labeled with the number inside the box you increased;

(iii) continue (i) and (ii) until the tableau cannot be changed.

The corresponding pictures for $\lambda = 3\epsilon_1 + 2\epsilon_2 + \epsilon_3$ in $\mathfrak{gl}_3$ can be seen below.





Chapter 7

Quantum Affine $\mathfrak{sl}_2$ and Its Highest Weight Module $V(\Lambda_0)$

In this chapter we will define the quantum affine algebra of the special linear Lie algebra and discuss the crystal structure of the *fundamental weight*, Λ_0. We will show how we can view the crystal graph structure in a new representation encodes the same behavior as the conventional notation, but allows us to better see more combinatorial patterns that arise in the crystal structure of the highest weight vector. Again, we rely heavily on notation used by [HK02, Ch.9].

7.1 The quantum affine algebra $\mathfrak{U}_q(\widehat{\mathfrak{sl}}_2)$

Let $I = \{0, 1\}$ be the the index set and A be the generalized Cartan matrix $\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$ of affine $A_1^{(1)}$ type. Moreover, let $\Pi = \{\alpha_0, \alpha_1\}$, $\Pi^\vee = \{h_0, h_1\}$, and $P^\vee = \mathbb{Z}h_0 \oplus \mathbb{Z}h_1 \oplus \mathbb{Z}d$, where d is defined as $\alpha_0(d) = 1,\ \alpha_1(d) = 0$.

Definition 7.1.1. A **fundamental weight** Λ_i, where $i \in I$, is a linear functional on $\mathfrak{h} = \mathbb{C} \otimes_{\mathbb{Z}} P^\vee$ that satisfies the identities $\Lambda_i(h_j) = \delta_{ij}$ and $\Lambda_i(d) = 0$.

From this, we also fix some additional notation. Namely, the *null root* is defined to be $\delta = \alpha_0 + \alpha_1$ such that the weight lattice, P, is $P = \mathbb{Z}\Lambda_0 \oplus \mathbb{Z}\Lambda_1 \oplus \mathbb{Z}\delta$. So, the **quantum affine algebra** $\mathfrak{U}_q(\widehat{\mathfrak{sl}}_2)$ is the quantum group with the corresponding Cartan datum $(A, \Pi, \Pi^\vee, P, P^\vee)$ of affine type $A_1^{(1)}$. For brevity, we will denote the subalgebra of $\mathfrak{U}_q(\widehat{\mathfrak{sl}}_2)$ generated by $e_i,\ f_i,\ q^{\pm h}$ as $\mathfrak{U}'_q(\widehat{\mathfrak{sl}}_2)$. We will also, by convention, call $\mathfrak{U}'_q(\widehat{\mathfrak{sl}}_2)$ the quantum affine algebra of type $A_1^{(1)}$. Therefore, $\mathfrak{U}'_q(\widehat{\mathfrak{sl}}_2)$ is a quantum group with the Cartan datum $(A, \Pi, \Pi^\vee, \bar{P}, \bar{P}^\vee)$, where the $\delta = 0$ in $\bar{P}$ and $\bar{P}^\vee = \mathbb{Z}h_0 \oplus \mathbb{Z}h_1$. The elements of P and $\bar{P}$ are called **affine weights** and **classical weights** respectively. The reader may note that we are using the classical limit notation by placing a bar over the image of the classical projection. To this end, we consider an embedding aff : $P \to \bar{P}$ such that the composition $\mathrm{cl} \circ \mathrm{aff} = \mathrm{id}_{\bar{P}}$ and $\mathrm{aff} \circ \mathrm{cl}(\alpha_i) = \alpha_i$.

Remark 18. [HK02, Ch 9.2] An important aspect about $\mathfrak{U}_q(\widehat{\mathfrak{sl}}_2)$ and $\mathfrak{U}'_q(\widehat{\mathfrak{sl}}_2)$ is that $\mathfrak{U}'_q(\widehat{\mathfrak{sl}}_2)$ allows for finite dimensional irreducible modules, as where any non-trivial irreducible $\mathfrak{U}_q(\widehat{\mathfrak{sl}}_2)$-module is infinite dimensional. However, infinite dimensional irreducible $\mathfrak{U}_q(\widehat{\mathfrak{sl}}_2)$-modules have finite dimensional weight spaces, whereas the weight spaces of infinite dimensional $\mathfrak{U}_q'(\widehat{\mathfrak{sl}}_2)$-modules are infinite dimensional.

Let V be a finite dimensional $\mathfrak{U}_q'(\widehat{\mathfrak{sl}}_2)$-module and let z be an indeterminate such that $V^{\mathrm{aff}} = \mathbb{C}[z, z^{-1}] \otimes_{\mathbb{C}} V$. We define the action of $\mathfrak{U}_q(\widehat{\mathfrak{sl}}_2)$ on V^{aff} to be

$$\begin{aligned} e_1(z^m \otimes v) &= z^m \otimes e_1 v, & e_0(z^m \otimes v) &= z^{m+1} \otimes e_0 v, \\ f_1(z^m \otimes v) &= z^m \otimes f_1 v, & f_0(z^m \otimes v) &= z^{m+1} \otimes f_0 v, \\ q_1^{h_1}(z^m \otimes v) &= z^m \otimes e_1 v, & q_0^{h_0}(z^m \otimes v) &= z^m \otimes q_0^{h_0} v, \\ q^d(z^m \otimes v) &= q^m z^m \otimes v. \end{aligned}$$

We call the $\mathfrak{U}_q(\widehat{\mathfrak{sl}}_2)$-module V^{aff} defined this way the **affinization** of V. Let ζ be a nonzero complex number. Again, as when we though about the crystal limit, we will consider the maximal ideal of $\mathbb{C}[z, z^{-1}]$ generated by $z - \zeta$, denoted $\mathbf{J}_\zeta$. To this end, we achieve the isomorphism $\mathbb{C} \cong \mathbb{C}[z, z^{-1}]/\mathbf{J}_\zeta$. This allows us to define the **evaluation module** V_ζ of V where

$$V_\zeta = \mathbb{C} \otimes_{\mathbb{C}[z,z^{-1}]} V^{\mathrm{aff}} \cong V^{\mathrm{aff}}/\mathbf{J}_\zeta V^{\mathrm{aff}}$$

is a $\mathfrak{U}_q'(\widehat{\mathfrak{sl}}_2)$-module.

Let $V = \mathbb{C}v_1 \oplus \mathbb{C}v_{-1}$ be $\mathfrak{U}_q'(\widehat{\mathfrak{sl}}_2)$-module defined by

$$\begin{aligned} e_1 v_1 &= 0, & f_1 v_1 &= v_{-1}, & q^{h_1} v_1 &= q v_1, \\ e_0 v_1 &= v_{-1}, & f_0 v_1 &= 0, & q^{h_0} v_1 &= q^{-1} v_1, \\ e_1 v_{-1} &= v_1, & f_1 v_{-1} &= 0, & q^{h_1} v_{-1} &= q^{-1} v_{-1}, \\ e_0 v_{-1} &= 0, & f_0 v_{-1} &= v_1, & q^{h_0} v_{-1} &= q v_{-1}. \end{aligned}$$

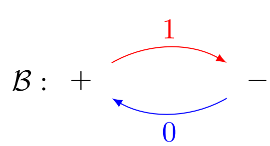
We can define the affinization of V and construct the evaluation module V_ζ, which has the action defined by

$$\begin{aligned} e_1 v_1 &= 0, & f_1 v_1 &= v_{-1}, & q^{h_1} v_1 &= q v_1, \\ e_0 v_1 &= \zeta v_{-1}, & f_0 v_1 &= 0, & q^{h_0} v_1 &= q^{-1} v_1, \\ e_1 v_{-1} &= v_1, & f_1 v_{-1} &= 0, & q^{h_1} v_{-1} &= q^{-1} v_{-1}, \\ e_0 v_{-1} &= 0, & f_0 v_{-1} &= \zeta^{-1} v_1, & q^{h_0} v_{-1} &= q v_{-1}. \end{aligned}$$

For further elaboration on this module, see Example 9.2.2 in [HK02, Ch. 9.2].

7.2 Crystal Structure of Λ_0

Let $V = \mathbb{C}v_{-1} \oplus \mathbb{C}v_1$ be the two $\mathfrak{U}_q'(\widehat{\mathfrak{sl}}_2)$-module as defined above. Then if we let $v_{-1} \equiv +$ and $v_1 \equiv -$, then the crystal graph of V becomes the following figure.

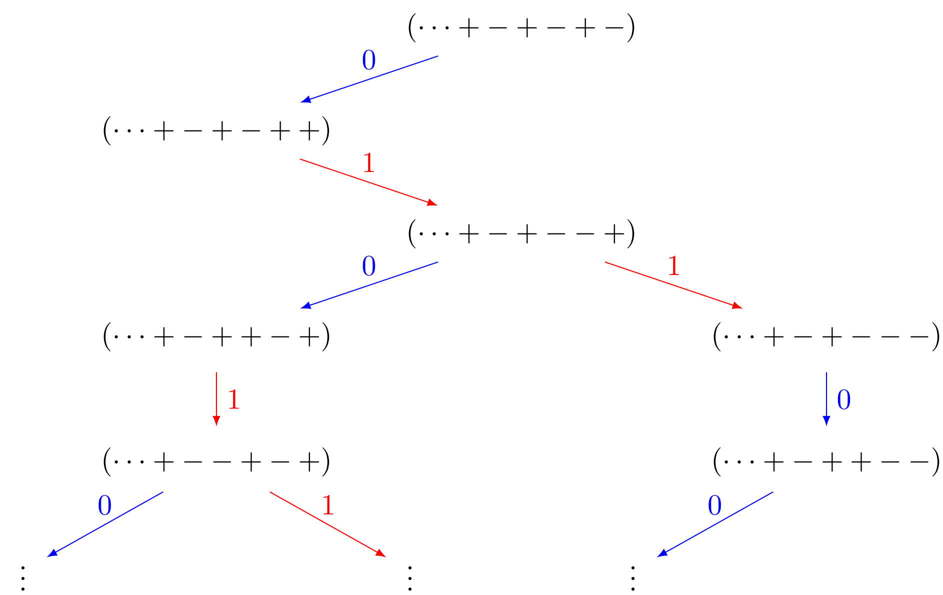


We can see that in contrast to previous crystal graphs in the finite case, there is a possibility of an infinite string of alternating pluses and minuses here in the affine case. We wish to begin by looking at the highest weight vector u_{Λ_0} in $\mathcal{B}(\Lambda_0)$, which can be identified by the infinite sequence $(\cdots + - + - + -)$. For verification of this, see [HK02, Ch 9.3]. The crystal structure of Λ_0, can be created using several rules for determining how to act using the Kashiwara operators.

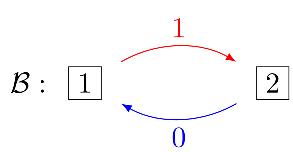
1. For the action of f_1 we cancel out all plus-minus pairs, going from left to right, and act on the left most $+$, changing the $+$ to a $-$.

2. For the action of f_0 we cancel out all plus-minus pairs, going from right to left, and act on the left most $-$, changing the $-$ to a $+$.

Using these rules, we have the crystal structure of $\mathcal{P}(\Lambda_0)$ below.



Now we wish to alter the notation used. Begin my making the following equivalencies: let a plus be denoted by $\boxed{1}$, a minus be denoted by $\boxed{2}$, and let $(\cdots + - + - + -)$ be denoted by $\boxed{\bullet}$. The new crystal graph can be seen in the following figure.



With this change in notation, the action of f_0 and f_1 go essentially unchanged, but realized in the following way.

1'. For the action of f_1 remove all $\boxed{1} \otimes \boxed{2}$ and turn the left most $\boxed{1}$ to a $\boxed{2}$.

2'. For the action of f_0 remove all $\boxed{2} \otimes \boxed{1}$ and turn the left most $\boxed{2}$ to a $\boxed{1}$.

Using this notation, we can see some patterns occur. Consider the following example.

Example 7.2.1. We wish to show that the action of f_0 turns $\boxed{\bullet}$ into $\boxed{\bullet}\otimes\boxed{1}\otimes\boxed{1}$. Begin by realizing that $\boxed{\bullet} = \boxed{\bullet}\otimes\boxed{1}\otimes\boxed{2}$. By the action of f_0, we cancel all $\boxed{2}\otimes\boxed{1}$ pairs and result in the cancellation of all but the very last $\boxed{2}$, which would be turned into a $\boxed{1}$ by the action of f_0. Thus we have shown what we wish to show.

Remark 19. The importance of this example lies not in the computation, but the realization that we can manipulate the $\boxed{\bullet}$ in such a way that we can realize the action of f_0 and f_1 in the same way we did with the sequence of plus-minuses, but now we are doing so in a less intimidating way (i.e void of the appearance of infinite length).

If we translate the previous graph of $\mathcal{P}(\Lambda_0)$ using this new notation, we arrive at the following figure.

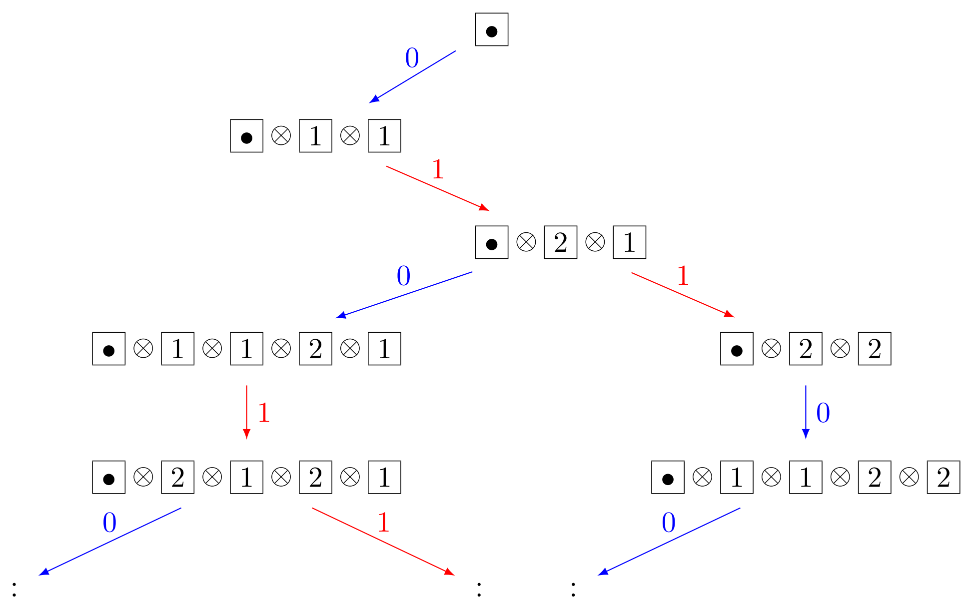


Figure 7.1

7.3 Analysis of $V(\Lambda_0)$

In this section we will analyze some features of Λ_0 including the number of weights that occur in each level and the multiplicities of weights in a particular level. To compute this, the reader may reference the provided code supplied with this text. The code is written in Maple; therefore, access to Maplesoft's Maple software is required to run the program. However, due to the simplicity of the code, the program could be easily rewritten in any open source language.

In Appendix C, the reader can find a table providing some of the information the program provides organized by levels. The first column of the table is the level with the enumeration starting at zero; the second column is the number of elements in the corresponding level; the third level is the number of elements that have resulted from i zero arrows and j one arrows. This is represented by $\beta \times [i.j]$, $\beta \in \mathbb{N}$, which means that there are β elements that are the result of applying i zero arrows and j one arrows to Λ_0 and thus are of weight $\Lambda_0 - i\alpha_0 - j\alpha_1 = \Lambda_0 - 2(j-i)\Lambda_0 + 2(i-j)\Lambda_1 - i\delta$.

Some patterns in these elements become strikingly obvious with the familiarity of integer sequences. We begin by looking at the number of elements in a particular level. These numbers, as I have stated, can be found in the second column of the table in Appendix C. The importance of starting the level enumeration at zero becomes necessary here.

Definition 7.3.1. Let $p(n)$ be the partitioning function that gives the number of ways of writing the integer n as a sum of positive integers without regard to order. Let $q(n)$ be the partitioning function that gives the number of ways of writing the integer n as a sum of positive integers without regard to order with the restriction that all integers in a given partition are distinct.

If we consider the sequence of number of elements in a given partition, then we see that the sequence up to 12 levels is $1, 1, 1, 2, 2, 3, 4, 5, 6, 8, 10, 12, 15, \ldots$, which we can observe as the output of $q(n)$, where n starts at zero. This leads us to our first conjecture.

Conjecture 1. *The number of elements at the nth level in $\mathcal{P}(\Lambda_0)$ is exactly $q(n)$.*

Now we consider the even levels, written in the form $2n$. If we write the most abundant weight multiplicities, then we get the sequence $1, 1, 2, 3, 5, 7, 11, 15, 22, \ldots$, which is the output of $p(n)$, with n starting at zero. Using the notation from the table in Appendix C, we see that each of these multiplicities is attached to an element of the form $[n, n]$. Moreover, the next most abundant weight multiplicity for the even level consist of the following sequence starting at level 6 ($= 2 \cdot 3$): $1, 1, 2, 3, 5, 7, 11, 15, 22, \ldots$, which is again $p(n)$, with n starting at 3. So, if we scale back n by three units, we can write this as $p(n-3)$ to ensure the counting starts at n corresponding to the sixth level. Attached to these multiplicities are elements of the form $[n+1, n-1]$.

We wish to shift our focus to the odd level, written in the form $2n + 1$. If we write the most abundant weight multiplicities, then we again get the sequence of $p(n)$, with n starting at zero. Attached to these multiplicities are elements of the form $[n + 1, n]$. The second most abundant weight multiplicities also follow the sequence of $p(n)$, with n starting at 1. The elements attached to these multiplicities are of the form $[n, n+1]$. We summarize these two paragraphs in the following conjecture.

Conjecture 2. *Let n be an integer. The the following are true:*

(i) For a level $2n$, there are $p(n)$ elements which are generated by n actions of f_0 and n actions of f_1.

(ii) For a level $2n \geq 6$, there are $p(n-3)$ elements which are generated by $n+1$ actions of f_0 and $n-1$ actions of f_1.

(iii) For a level $2n + 1$, there are $p(n)$ elements which are generated by $n + 1$ actions of f_0 and n actions of f_1.

(iv) For a level $2n+1 \geq 1$, there are $p(n-1)$ elements which are generated by n actions of f_0 and $n+1$ actions of f_1.

Appendix A

Tensor Products and the Tensor Algebra

A.1 Tensor Product

Let V and W be a vector space over a field $\mathbb{F}$. We now construct a vector space $\mathcal{S}$ over $\mathbb{F}$ with the elements of the Cartesian product set $V \times W$ as basis vectors. Therefore, if $v \in V$ and $w \in W$, then any vector $u \in S$ can be written as a linear combination of elements in the form $\alpha_{v,w}(v, w)$, where $\alpha + v, w \in \mathbb{F}$ and only finitely many $\alpha_{v,w}$ are nonzero. Then we can consider the subspace of S spanned by all the vectors of the following form:

$$R = \begin{cases} (\alpha v_1 + \beta v_2, w) - [\alpha(v_1, w) + \beta(v_2, w)], \\ (v, \alpha w_1 + \beta w_2) - [\alpha(v, w_1) + \beta(v, w_2)], \end{cases}$$

where $v, v_1, v_2 \in V,\ w, w_1, w_2 \in W,\ \alpha, \beta \in \mathbb{F}$. Then the quotient space S/R, which we denote by $V \otimes W$, is defined to be the **tensor product** of V and W over $\mathbb{F}$. To this end, we can see that the following is true:

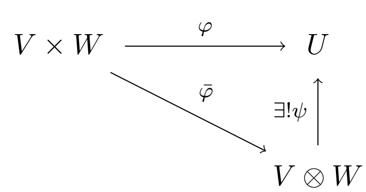
$$v_1 \otimes v_2 \otimes \alpha v_3 \otimes \cdots \otimes \beta v_{n-1} \otimes v_n = \alpha\beta(v_1 \otimes v_2 \otimes v_3 \otimes \cdots \otimes v_{n-1} \otimes v_n).$$

By this construction, we can extend this construction to finite sums of tensor products and create a multilinear "gadget" which we can use to construct another useful algebra.

A.1.1 Properties of Tensor Products

The following are several properties of tensor products that are known.

Theorem A.1.1. *If φ is a bilinear map from $V \times W$ to a vector space U, then there exists a unique linear transformation $\psi : V \otimes W \to U$ such that the following diagram commutes.*



Theorem A.1.2. *Let I and J be indexing sets. Moreover, let $\{v_i|\ i \in I\}$ be a basis for V and $\{w_j|\ j \in J$ be a basis for W. Then $\{v_i \otimes w_j|\ i \in I,\ j \in J\}$ is a basis for $V \otimes W$.*

Proofs of these theorems can be found in [DF03, Ch 11.2]

A.2 Tensor Algebra

A.2.1 Construction

Let V be a vector space over an arbitrary field $\mathbb{F}$. For any nonnegative integer k, we define the $\mathbf{k^{th}}$ **tensor power** of V to be the **tensor product** of V with itself k times to be

$$T^k V = V^{\otimes k} = V \otimes V \otimes V \otimes \cdots \otimes V.$$

From this, we can construct the **tensor algebra** as the direct sum of $T^k V$ for $k = 0, 1, 2 \ldots$. We denote this by $\mathcal{T}(V)$ and is written as

$$\mathcal{T}(V) = \bigoplus_{k=0}^{\infty} T^k V = \mathbb{F} \oplus V \oplus (V \otimes V) \oplus (V \otimes V \otimes V) \oplus \cdots.$$

The tensor algebra $\mathcal{T}$ is consider the *free algebra* in that it is the noncommutative analogue of a polynomial ring on the vector space V. The tensor algebra, like many concepts in abstract mathematics, satisfies the following universal property:

Let i be an algebra homomorphism of V into $\mathcal{T}(V)$. Any linear transformation $\varphi : V \to A$ to an algebra A over $\mathbb{F}$ can be uniquely extended to an algebra homomorphism, $\bar{\varphi}$, from $\mathcal{T}(V)$ to A. The mapping i is more specifically a canonical inclusion of V into $\mathcal{T}(V)$.

Moreover, we say that the following diagram commutes.

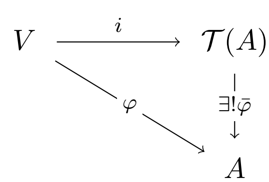
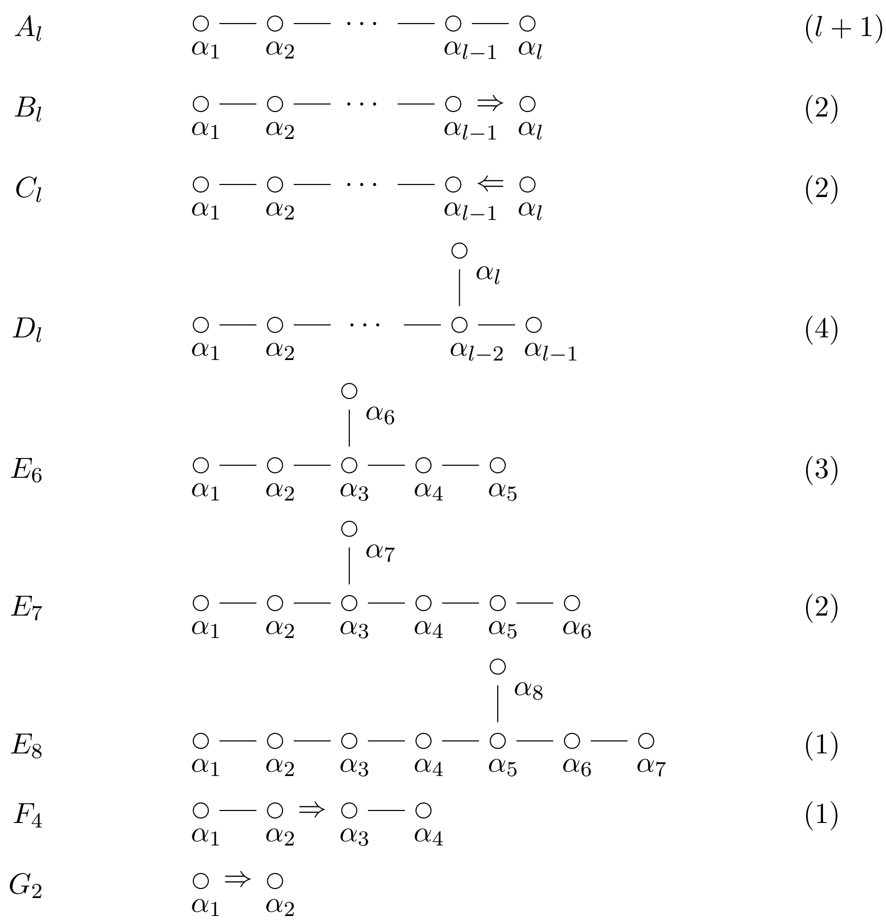


Figure A.1: Universal Property of $\mathcal{T}(V)$

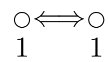
Appendix B

Dynkin Diagrams

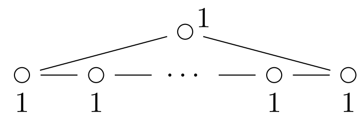
B.0.2 Classical Simple and Exceptional Lie Algebras



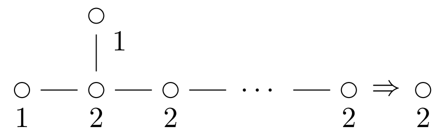
$A_1^{(1)}$



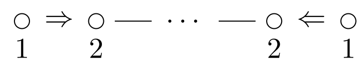
$A_l^{(1)} (l \geq 2)$



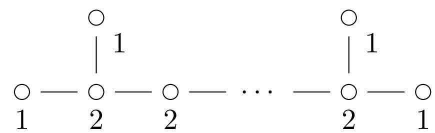
$B_l^{(1)} (l \geq 3)$



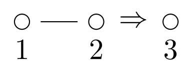
$C_l^{(1)} (l \geq 2)$



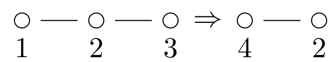
$D_l^{(1)} (l \geq 4)$



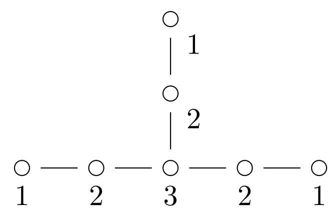
$G_2^{(1)}$



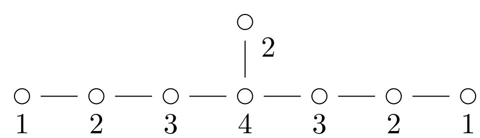
$F_4^{(1)}$



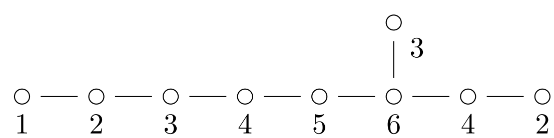
$E_6^{(1)}$



$E_7^{(1)}$



$E_8^{(1)}$



Appendix C

Output of Maple Code

Level	# Elements in Level	Weight Multiplicities
0	1	$1 \times [0,0]$
1	1	$1 \times [1,0]$
2	1	$1 \times [1,1]$
3	2	$1 \times [2,1]$ $1 \times [1,2]$
4	2	$2 \times [2,2]$
5	3	$2 \times [3,2]$ $1 \times [2,3]$
6	4	$3 \times [3,3]$ $1 \times [4,2]$
7	5	$3 \times [4,3]$ $2 \times [3,4]$
8	6	$5 \times [4,4]$ $1 \times [5,3]$
9	8	$5 \times [5,4]$ $3 \times [4,5]$
10	10	$7 \times [5,5]$ $2 \times [6,4]$ $1 \times [4,5]$
11	12	$7 \times [6,5]$ $5 \times [5,6]$
12	15	$11 \times [6,6]$ $3 \times [7,5]$ $1 \times [5,7]$
13	18	$11 \times [7,6]$ $7 \times [6,7]$
14	22	$15 \times [7,7]$ $5 \times [8,6]$ $2 \times [6,8]$

Level	# Elements in Level	Weight Multiplicities
15	27	$15 \times [8,7]$ $11 \times [7,8]$ $1 \times [9,6]$
16	32	$22 \times [8,8]$ $7 \times [7,8]$ $3 \times [7,9]$
17	38	$22 \times [9,8]$ $15 \times [8,9]$ $1 \times [10,7]$
18	46	$30 \times [9,9]$ $11 \times [10,8]$ $5 \times [8,10]$
19	54	$30 \times [10,9]$ $22 \times [9,10]$ $2 \times [11,8]$
20	64	$42 \times [10,10]$ $15 \times [11,9]$ $7 \times [9,11]$
21	76	$42 \times [11,10]$ $30 \times [10,11]$ $3 \times [12,9]$ $1 \times [9,12]$
22	89	$56 \times [11,11]$ $22 \times [12,10]$ $11 \times [10,12]$
23	104	$56 \times [12,11]$ $42 \times [11,12]$ $5 \times [13,10]$ $1 \times [10,13]$
24	122	$77 \times [12,12]$ $30 \times [13,11]$ $15 \times [11,13]$
25	142	$77 \times [13,12]$ $56 \times [12,13]$ $7 \times [14,11]$ $2 \times [11,14]$
26	165	$101 \times [13,13]$ $42 \times [14,12]$ $22 \times [12,14]$
27	192	$101 \times [14,13]$ $77 \times [13,14]$ $11 \times [15,12]$ $3 \times [12,15]$

Level	# Elements in Level	Weight Multiplicities
28	222	$135 \times [14, 14]$
		$56 \times [15, 13]$
		$30 \times [13, 15]$
		$1 \times [16, 12]$
29	256	$135 \times [15, 14]$
		$101 \times [14, 15]$
		$15 \times [16, 13]$
		$5 \times [13, 16]$
30	296	$176 \times [15, 15]$
		$77 \times [16, 14]$
		$42 \times [14, 16]$
		$1 \times [17, 13]$
31	340	$176 \times [16, 15]$
		$135 \times [15, 16]$
		$22 \times [17, 14]$
		$7 \times [14, 17]$
32	390	$231 \times [16, 16]$
		$101 \times [17, 15]$
		$56 \times [15, 17]$
		$2 \times [18, 14]$
33	448	$231 \times [17, 16]$
		$176 \times [16, 17]$
		$30 \times [18, 15]$
		$11 \times [15, 18]$
34	512	$297 \times [17, 17]$
		$135 \times [18, 16]$
		$77 \times [16, 18]$
		$3 \times [19, 15]$
35	585	$297 \times [18, 17]$
		$231 \times [17, 18]$
		$42 \times [19, 16]$
		$15 \times [16, 19]$
36	668	$385 \times [18, 18]$
		$176 \times [19, 17]$
		$101 \times [17, 19]$
		$5 \times [10, 16]$
		$1 \times [16, 20]$
37	760	$385 \times [19, 18]$
		$297 \times [18, 19]$
		$56 \times [20, 17]$
		$22 \times [17, 20]$
38	864	$490 \times [19, 19]$
		$231 \times [20, 18]$
		$135 \times [18, 20]$
		$7 \times [21, 17]$
		$1 \times [17, 21]$

Bibliography

[BP02] Stephen Berman and Karen Hunger Parshall. Victor kac and robert moddy: Their paths to kac-moody lie algebras. *The Mathemamtical Intelligencer*, 24:50–60, 2002.

[DF03] David Dummit and Richard Foote. *Abstract Algebra*. John Wiley & Sons, Inc., 3 edition, 2003.

[EW06] Karin Erdmann and Mark J. Wildon. *Introduction to Lie Algebras*. Springer, 2006.

[HK02] Jin Hong and Seok-Jin Kang. *Introduction to Quantum Groups and Crystal Bases*. American Mathematical Society, 1 edition, 2002.

[Hum97] James Humpherys. *Introduction to Lie Algebras and Representation Theory*. Springer-Verlag, 8 edition, 1997.

[Jac79] Nathan Jacobson. *Lie Algebra*. Dover Publications, 1 edition, 1979.

[Kac94] Victor Kac. *Infinite-dimensional Lie Algebras*. Cambridge University Press, 3 edition, 1994.

[Kas95] Masaki Kashiwara. On crystal bases. In *Canadian Mathematical Socieity Proceedings*, March 1995.

[Lus90] George Lusztig. Finite dimenstional hopf algebras arising from quantized universal enveloping algebras. *Journal of the American Mathematical Socieity*, 3:257–296, 1990.